FLORENTIN SMARANDACHE

SPECIAL PSEUDO LINEAR ALGEBRAS USING [0,n)

Special Pseudo Linear Algebras using [0,n)

W. B. Vasantha Kandasamy
Florentin Smarandache

Educational Publisher Inc.
Ohio
2014

CONTENTS

Preface 5

Chapter One
INTRODUCTION 7

Chapter Two
SPECIAL PSEUDO LINEAR ALGEBRAS USING THE INTERVAL [0, n) 9

Chapter Three
SMARANDACHE SPECIAL INTERVAL PSEUDO LINEAR ALGEBRAS 107

Chapter Four
SMARANDACHE STRONG SPECIAL PSEUDO INTERVAL VECTOR SPACES 223

FURTHER READING 266

INDEX 268

ABOUT THE AUTHORS 270

PREFACE

In this book we introduce some special type of linear algebras called pseudo special linear algebras using the interval [0, n). These new types of special pseudo interval linear algebras has several interesting properties. Special pseudo interval linear algebras are built over the subfields in Z_n where Z_n is a S-ring. We study the substructures of them.

The notion of Smarandache special interval pseudo linear algebras and Smarandache strong special pseudo interval linear algebras are introduced. The former S-special interval pseudo linear algebras are built over the S-ring itself. Study in this direction has yielded several interesting results.

S-strong special pseudo interval linear algebras are built over the S-pseudo interval special ring [0, n). SSS-pseudo special linear algebras are mainly introduced for only on these new structures, study, develop, describe and define the notion of SSS-linear functionals, SSS-eigen values, SSS-eigen vectors and SSS-polynomials.

This type of study is important and interesting. Authors are sure these structures will find applications as in case of usual linear algebras.

The authors wish to acknowledge Dr. K Kandasamy for his sustained support and encouragement in the writing of this book.

W.B.VASANTHA KANDASAMY
FLORENTIN SMARANDACHE

Chapter One

INTRODUCTION

In this chapter we recall the operations on the special interval $[0, n)$, $n < \infty$ where addition and multiplication are performed modulo n. This is a special study for $Z_n \subseteq [0, n)$ and $[0, n)$ can be realized as the real closure of Z_n.

$[0, n)$ is never a group under product only a semigroup. Thus if n is a prime $[0, n)$ happens to be an infinite pseudo integral domain.

Three types of vector spaces and constructed using $[0, n)$. This study is new and innovative. For always Z_n is imagined to be a ring (a field in case n is a prime) with only n number of elements in them. However $[0, n)$ has infinite number of elements in them.

When $[0, n)$ is used (n a prime) then we have usual vector spaces constructed using $[0, n)$ over the field Z_p.

The next stage of study being S-special interval vector spaces using $[0, n)$ over the S-ring Z_n. This will have meaning only if Z_n is a Smarandache ring. Finally we define the new notion of Smarandache Strong Special pseudo vector space (SSS-pseudo vector space) over the S-special pseudo interval ring [0, n).

Only these happen to pave way for finite dimensional vector space using [0, n). Further only using this structure one can define the notion of SSS-linear functionals and the SSS-dual space.

Such type of study is carried out for the first time.

This study will certainly lead to several new algebraic structure inventions.

Chapter Two

SPECIAL PSEUDO LINEAR ALGEBRAS USING THE INTERVAL [0, n)

In this chapter authors for the first time define special interval vector spaces defined over the S-rings. It is important to keep on record that [0, n) can maximum be a pseudo integral domain in case n is a prime and is Smarandache pseudo special interval ring whenever $Z_n \subseteq [0, n)$ is a S-ring.

Thus all special vector spaces built using [0, n) and special vector linear algebras built using [0, n) are only S-linear algebras unless we make, these structures over fields contained in [0, n). We will develop and describe them in this chapter.

DEFINITION 2.1: *Let* $V = \{[0, n), +\}$ *be an additive abelian group.* $F \subseteq Z_n$ *be a field so that* Z_n *is a Smarandache ring if n is a prime* $F = Z_p$ *is a field; we define V to be the special interval vector space over the field* $F \subseteq Z_n \subseteq [0, n)$*. Further we do not demand* $(a + b)v = av + bv$ *and* $a(v_1+v_2) = av_1 + av_2$ *for a, b* $\in F \subseteq Z_n$ *and* $v, v_1, v_2 \in V$*. The only criteria is* $av = va \in V$ *for all* $a \in F \subseteq Z_n$ *and* $v \in V$*.*

Note: V can be an additive abelian group built using [0, n) that is the only criteria for the construction of special interval vector spaces. The distributive laws may or may not be true.

Example 2.1: Let V = {[0, 5), +} be the special interval vector space over Z_5; the field of modulo integers.

Example 2.2: Let V = {[0, 6), +} be a special interval vector space over the field F = {0, 2, 4} $\subseteq Z_6 \subseteq$ [0, 6) or over the field F_1 = {0, 3} $\subseteq$ [0, 6).

Example 2.3: Let V = {[0, 15), +} be the special interval vector space over the field F = {0, 3, 6, 9, 12} $\subseteq Z_{15}$.

Let 0.315 $\in$ V then for a = 6 we have a × 0.315 = 1.890 $\in$ V.

Let a = 3 and b = 9 $\in$ V. For v= 6.021 $\in$ V we have

$$\begin{aligned}(a + b)\, v &= (3 + 9)v \\ &= 12 \times 6.021 = 72.252 \\ &= 12.252 \qquad \ldots (1)\end{aligned}$$

$$\begin{aligned}av + bv &= 3 \times 6.021 + 9 \times 6.021 \\ &= 18.063 + 54.189 \\ &= 72.252 \\ &= 12.252 \qquad \ldots (2)\end{aligned}$$

(1) and (2) are identical in this case hence (a + b) v = av + bv in V as a special interval vector space over F.

Let v_1 = 2.615 and v_2 = 7.215 $\in$ V and a = 6 $\in$ F.

We find

$$\begin{aligned}a(v_1 + v_2) &= 6\,(2.615 + 7.215) = 6\,(9.830) = 58.980 \\ &= 13.980 \qquad \ldots \text{I}\end{aligned}$$

$$\begin{aligned}av_1 + av_2 &= 6 \times 2.615 + 6 \times 7.215 \\ &= 15.690 + 43.290 \\ &= 58.980 \\ &= 13.980 \qquad \ldots \text{II}\end{aligned}$$

I and II are identical in this case hence this set of vectors in V distribute over the scalar, however one do not demand this condition in our definition.

$0.v = 0$ for all $v \in V$ and $a.0 = 0$ for all $a \in F$.

The following observations are important

1. Always the cardinality of V is infinite.
2. The advantage when n is a composite number and if Z_n is S-ring we can have more than one special interval vector space.

The number of spaces depends on the number of fields the ring Z_n has.

Example 2.4: Let $V = \{[0, 12), +\}$ be a special interval vector space over the field $F_1 = \{0, 4, 8\}$. We see this special interval vector space can be defined only over one field. For Z_{12} has only one subset which is a field.

Example 2.5: Let $V = \{[0, 24), +\}$ be a special interval vector space over the field $F_1 = \{0, 8, 16\} \subseteq Z_{24}$. This is only special interval vector space of the interval [0, 24) over the field in Z_{24}.

Example 2.6: Let $V_1 = \{[0, 30), +\}$ be a special interval vector space over the field $F_1 = \{0,15\} \subseteq Z_{30}$.

$V_2 = \{[0,30), +\}$ be a special interval vector space over the field $F_2 = \{0, 10, 20\} \subseteq Z_{30}$.

$V_3 = \{[0, 30), +\}$ be the special interval vector space over the field $F_3 = \{0, 6, 12,18, 24\} \subseteq [0, 30)$; we have only three vector spaces over the three fields in $Z_{30} \subseteq [0, 30)$.

Example 2.7: Let $V = \{[0, 23), +\}$ be a special interval vector space over the field Z_{23}. We have a unique special interval vector space over the field $F = Z_{23}$.

Example 2.8: Let V = {[0, 29), +} be a special interval vector space over the field Z_{29} = F.

Example 2.9: Let V_1 = {[0, 143), +} be a special interval vector space over the field Z_{143} = F.

Example 2.10: Let V = {[0, 43), +} be a special interval vector space over the field F = Z_{43}.

In view of all this we have the following theorems.

THEOREM 2.1: *Let V = {[0, p), +} be the special interval vector space over the field Z_p = F (p a prime); V is the only one special interval vector space over Z_p.*

Proof is direct hence left as an exercise to the reader.

THEOREM 2.2: *Let V = {[0, n), +} be the special interval vector space over t fields, $F_i \subseteq Z_n$, $1 \leq i \leq t$, hence using this V we have t distinct special interval vector spaces over each of the fields, F_i, ($1 \leq i \leq t$).*

Proof : If Z_n is a S-ring and Z_n has t number of subsets F_i such that each F_i is a field then we have V = {[0, n), +} to be special interval vector space over F_i for i = 1, 2, …, t.

Hence the claim.

We will illustrate this situation by some examples.

Example 2.11: Let V ={[0, 42), +} be a special interval vector space over the field F_1= {0, 21} $\subseteq Z_{42}$.

Let F_2 ={0, 14, 28} $\subseteq Z_{42}$ be a field in Z_{42}.

V_2 ={[0, 42), +} is a special interval vector space over the field F_2.

Let $F_3 = \{0, 6, 12, 18, 24, 30, 36\} \subseteq Z_{42}$ be the field. $(F_3 \setminus \{0\}, \times\}$ is given by the following table

×	6	12	18	24	30	36
6	36	30	24	18	12	6
12	30	18	6	36	24	12
18	24	6	30	12	36	18
24	18	36	12	36	6	24
30	12	24	36	6	18	30
36	6	12	18	24	30	36

36 acts as the identity with respect to multiplication of the field $F_3 = \{0, 6, 12, 18, 24, 30, 36\} \subseteq Z_{42}$.

We have three different special interval vector spaces.

For if $0.65 \in V$ now V as a special interval vector space over F_1 we get

$0.65 \times 21 = 13.65 \in V$.

Now $0.65 \times 14 = 9.10 \in V$ and $0.65 \times 30 = 19.50 \in V$.

We see the three spaces are distinct.

Now we proceed onto discuss about special interval vector subspaces of a special interval vector spaces.

Example 2.12: Let $V = \{[0, 15), +\}$ be the special interval vector space over the field $F = \{0, 5, 10\} \subseteq Z_{15}$.

$P_1 = \{0, 1, 2, 3, \ldots, 14\} \subseteq V$ is a special interval vector subspace of V over F.

$P_2 = \{0, 3, 6, 9, 12\} \subseteq V$ is also a special vector subspace of V over F.

Example 2.13: Let V = {[0, 7), +} be a special interval vector space over the field F = Z_7 = {0, 1, 2, …, 6}.
P = {0, 1, 2, 3, 4, 5, 6} ⊆ V is a subspace of V over F = Z_7.

P_1 = {0. 0.5, 1, 1.5, 2, 2.5, 3, 3.5, …, 6.5} ⊆ V is a special interval vector subspace of V over Z_7.

Example 2.14: Let V = {[0, 3), +} be a special interval vector space over the field F = {0, 1, 2} = Z_3. P_1 = {0, 1, 2} ⊆ V is a subspace of V over F. P_2 = {0, 0.5, 1, 1.5, 2, 2.5} ⊆ V is again a subspace of V over F.

P_3 = {0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6, 1.8, 2, 2.2, 2.4, 2.6, 2.8} ⊆ V is also a subspace of V over F.

P_4 = {0, 0.25, 0.5, 0.752, 1, 1.25, 1.50, 1.75, 2, 2.25, 2.5, 2.75} ⊆ V is also a subspace of V over F.

Example 2.15: Let V = {[0, 12), +} be the special interval vector space over the field F = {0, 4, 8}. P_1 = {0, 1, 2, …, 10, 11} ⊆ V is a special interval vector subspace of V over F.

P_2 = {0, 6} ⊆ V is a special interval vector subspace of V over F.

P_3 = {0, 3, 6, 9} ⊆ V is a special interval vector subspace of V over F.

P_4 = {0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, …, 10, 10.5, 11, 11.5} ⊆ V is a special interval vector subspace of V over F.

P_5 = {0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, …, 1.8, 2, 2.2, 2.4, …, 10, 10.2, 10.4, 10.6, 10.8, 11, 11.2, 11.4, 11.6, 11.8} ⊆ V is a special interval vector subspace of V over F.

Example 2.16: Let V = {[0, 10), +} be the special interval vector space over the field F = {0, 2, 4, 6, 8} ⊆ Z_{10}.

$P_1 = \{0, 5\} \subseteq V$ is a subspace of V over F.

$P_2 = \{0, 1, 2, \ldots, 9\} \subseteq V$ is a subspace of V over F.

$P_3 = \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 9, 9.5\} \subseteq V$ is a subspace of V over F.

$P_4 = \{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, \ldots, 9, 9.2, 9.4, 9.6, 9.8\} \subseteq V$ is again a subspace of V over F.

Z_{10} has only two subsets which are fields; viz $F = \{0, 5\}$ and $F_1 = \{0, 2, 4, 6, 8\}$.

Example 2.17: Let $V = \{[0, 60), +\}$ be a special interval vector space over the field $F = \{0, 20, 40\} \subseteq Z_{60}$.

$P_1 = \{0, 1, 2, \ldots, 59\} \subseteq V$ is a special interval vector subspace of V over F.

$P_2 = \{0, 2, 4, 6, 8, \ldots, 58\} \subseteq V$ is again a special interval subspace of V over F.

$P_3 = \{0, 3, 6, 9, 12, \ldots 57\} \subseteq V$ is also special interval subspace of V over F.

$P_4 = \{0, 6, 12, 18, 24, 30, \ldots, 54\} \subseteq V$ is again a special interval subspace of V over F.

$P_5 = \{0, 10, 20, 30, 40, 50\} \subseteq V$ is a special interval subspace of V over F.

$P_6 = \{0, 15, 30, 45\} \subseteq V$ is also a special interval subspace of V over F.

$P_7 = \{0, 12, 24, 36, 48\} \subseteq Z_{60}$ is again a special interval subspace of V over F.

$P_8 = \{0.5, 0, 1, 1.5, 2, 2.5, \ldots, 58.5, 59, 59.5\} \subseteq Z_{60}$ is also a subspace of V over F.

Infact we chose $F_1 = \{0, 12, 24, 36, 48\}$ as a field then also we can get subspaces of that V over F_1.

Example 2.18: Let $V = \{[0, 73), +\}$ be a special interval vector space over the field Z_{73}. This V has subspaces given by $M_1 = \{0, 1, 2, \ldots, 73\} \subseteq V$ is a subspace of V over Z_{73}.

$M_2 = \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 72, 72.5\} \subseteq V$ is again a subspace of V over Z_{73}.

$M_3 = \{0, 0.125, 0.250, 0.375, 0.5, 0.625, \ldots, 72.125, 72.250, 72.375, 72.5, 72.625, 72.750, 72.875\} \subseteq V$ is again a subspace of V over Z_{73}. Even if in [0, p), p is a prime we see all special interval spaces have subspaces.

THEOREM 2.3: *Let $V = \{[0, p), +\}$ be a special interval vector space over the field Z_p (p a prime); V has several subspaces.*

Proof is direct hence left as an exercise to the reader.

Corollary : If [0, p) in theorem 2.3 p is replaced by n; n a composite number still $V = \{[0, n), +\}$ has several subspaces.

It is important and interesting to note all special interval vector spaces V defined using the interval [0, n) is such that $|V| = \infty$.

We will proceed onto define the concept of linear dependence and linear independence and a basis of V over the field $F \subseteq [0, n)$.

Let $V = \{[0, 27), +\}$ be a special vector space over the field $F = \{0, 9, 18\} \subseteq [0, 27)$.

Take $x = 2.04$ and $y = 3.3313 \in V$ we see 2.04 and 3.3313 are linearly independent for this x and y are not related by any scalar from F.

Let $x = 2.01$ and $y = 18.09 \in V$ we see $y = 9x$ so y and x are linearly dependent in V over the field F.

We can have several such concepts. We say a set of elements $B = \{v_1, v_2, \ldots, v_n\} \in V$ is a linearly independent set in V, if no v_i can be expressed in terms of v_j's $i \neq j$, j, i = 1, 2, …, n. That is $v_i \neq \sum \alpha_i v_j$ where α_i's $\in F$ and $v_j \in B$.

We say the linearly independent set B is said to be a basis if B generates V.

Recall here also we say

$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$ is possible if and only if each $\alpha_i = 0$. The authors feel that V has only infinite subset of V to be a basis.

That is V cannot have a finite basis over the field $F \subseteq [0, n)$.

Example 2.19: Let $V = \{[0, 5),. +\}$ be a vector space over the field $F = \{0, 1, 2, 3, 4\} = Z_5 \subseteq [0, 5)$. V is an infinite dimensional vector space over F.

V cannot have a finite basis.

However if $P_1 = \{0, 1, 2, 3, 4\} \subseteq [0, 5) \subseteq V$ be a special interval vector subspace of V then P_1 has dimension 1 over F. {1} or {2} or {3} or {4} is a basis of P_1 over F.

Let $P_2 = \{0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5\}$ be vector subspace of V over F.

$B = \{0.5\} \subseteq P_2$ is a basis of V over F. Thus dimension of P_2 over F is one.

Example 2.20: Let $V = \{[0, 13), +\}$ be a special interval vector space over the field Z_{13}.

V is an infinite dimensional vector space over Z_{13}.

V has also subspaces which are finite dimensional.

Next we proceed onto study the special interval vector spaces built using the interval group $G = \{[0, n), +\}$.

Example 2.21: Let $V = \{(a_1, a_2, a_3) \mid a_i \in [0, 15); 1 \leq i \leq 3, +\}$ be the special interval vector space over the field $F = \{0, 5, 10\} \subseteq Z_{15}$. V has infinite special interval vector subspaces also.

For take $P_1 = \{(a_1, 0, 0) \mid a_1 \in [0, 15), +\} \subseteq V$,

$P_2 = \{(0, a_1, 0) \mid a_1 \in [0, 15), +\} \subseteq V$,
$P_3 = \{(0, 0, a_1) \mid a_1 \in [0, 15), +\} \subseteq V$,
$P_4 = \{(0, a_1, a_2) \mid a_1, a_2 \in [0, 15), +\} \subseteq V$,
$P_5 = \{(a_1, a_2, 0) \mid a_1, a_2 \in [0, 15), +\} \subseteq V$ and
$P_6 = \{(a_1, 0, a_2) \mid a_1\ a_2 \in [0, 15), +\} \subseteq V$ are the six vector subspaces of V and $|P_i| = \infty$; $1 \leq i \leq 6$ and all of them are infinite dimensional over the field $F = \{0, 5, 10\} \subseteq Z_{15}$.

We have subspace M_i such that $|M_i| < \infty$. For take $M_1 = \{(a_1, 0, 0) \mid a_1 \in \{0, 5, 10\}, +\} \subseteq V$ is a subspace of V of dimension 1 over F.

$M_2 = \{(0, a_1, 0) \mid a_1 \in \{0, 3, 6, 9, 12\} \subseteq [0, 15)\} \subseteq V$ is also a subspace of V and $|M_2| < \infty$.

We can have atleast 14 such subspaces which has only finite number of elements in them.

Example 2.22: Let
$V = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in [0, 23), 1 \leq i \leq 5, +\}$ be the special interval vector space over the field $F = Z_{23}$.

V has both subspaces of finite dimension as well as infinite dimension.

$P_1 = (a_1, 0, 0, 0, 0) \mid a_1 \in [0, 23), +\} \subseteq V$ is a subspace of V of infinite dimension and $|P_1| = \infty$.

$P_2 = (a_1, 0, 0, 0, a_2) \mid a_1, a_2 \in [0, 23), +\} \subseteq V$ is also subspace of V over $F = Z_{23}$ and of infinite dimension.

$P_3 = (a_1, a_2, 0, 0, 0) \mid a_1, a_2 \in [0, 23), +\} \subseteq V$,
$P_4 = (0, 0, a_1, a_2, a_3) \mid a_1, a_2, a_3 \in [0, 23), +\} \subseteq V$,
$P_5 = (0, 0, 0, a_1, 0) \mid a_1 \in [0, 23), +\} \subseteq V$,
$P_6 = (a_1, a_2, a_3, a_4, 0) \mid a_i \in [0, 23), 1 \leq i \leq 4, +\} \subseteq V$,
$P_7 = (0, a_1, a_2, a_3, a_4) \mid a_i \in [0, 23), 1 \leq i \leq 4, +\} \subseteq V$,
$P_8 = (a_1, a_2, 0, a_3, a_4) \mid a_i \in [0, 23), 1 \leq i \leq 4, +\} \subseteq V$ and
$P_9 = (0, 0, a_1, a_2, 0) \mid a_i \in [0, 23), 1 \leq i \leq 2, +\}$ are all some of the subspaces of V over Z_{23}.

$M_1 = \{(a_1, 0, 0, 0, 0) \mid a_1 Z_{23}, +\} \subseteq V$,
$M_2 = \{(a_1, a_2, 0, 0, 0) \mid a_1, a_2 \in Z_{23}, +\} \subseteq V$,
$M_3 = \{(0, a_1, 0, 0, 0) \mid a_i \in Z_{23}, +\} \subseteq V$,
$M_4 = \{(0, 0, a_1, a_2, 0) \mid a_1, a_2 \in Z_{23}, +\} \subseteq V$ and
$M_5 = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in Z_{23}, +, 1 \leq i \leq 5\} \subseteq V$ are some of the subspaces of V over Z_{23}.

We see each M_i is such that $|M_i| < \infty$ and M_i's are finite dimensional over Z_{23}. For $1 \leq i \leq 3$.

Infact we have ${}_5C_1 + {}_5C_2 + {}_5C_3 + {}_5C_4 + 1$ number of subspace of finite order and finite dimensional over $F = Z_{23}$.

Example 2.23: Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} \middle| \, a_i \in [0, 24), 1 \leq i \leq 7, + \right\}$$

be a special interval vector space over the field F = {0, 8, 16} $\subseteq$ Z_{24}.

V has both finite and infinite dimensional vector subspaces.

Let

$$P_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \, a_1 \in [0, 24), + \right\} \subseteq V$$

be a special interval vector subspace over the F = {0, 8, 16} $\subseteq$ Z_{24}.

Clearly P_1 is infinite dimensional vector subspace of V over F.

Consider

$$M_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} \middle| \, a_i \in \{0, 8, 16\}, 1 \leq i \leq 7, + \right\} \subseteq V$$

is a special interval vector subspace of V over F.

Infact M_1 is finite dimensional over F and $|M_1| < \infty$.

$$M_2 = \left\{ \left[\begin{array}{c} a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_3 \end{array}\right] \middle| \; a_1, a_2, a_3 \in \{0, 8, 16\}, + \right\} \subseteq V$$

is a special interval vector subspace of V over F. M_2 is finite dimensional over F and $|M_2| < \infty$.

$$M_3 = \left\{ \left[\begin{array}{c} a_1 \\ 0 \\ a_2 \\ 0 \\ a_3 \\ 0 \\ a_4 \end{array}\right] \middle| \; a_i \in \{0, 8, 16\}, 1 \leq i \leq 4, + \right\} \subseteq V$$

is a special interval vector subspace of V over F.

M_3 is also finite dimensional over F and $|M_3| < \infty$.

$$M_4 = \left\{ \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \\ a_3 \end{array}\right] \middle| \; a_i \in \{0, 8, 16\}, 1 \leq i \leq 3, + \right\} \subseteq V$$

is a special interval vector subspace of V over F.

Clearly M_4 is also finite dimensional over F.

Let

$$B_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| a_1, a_2 \in [0, 24), + \right\} \subseteq V$$

be the special interval vector subspace of V over F.

B_1 is infinite dimensional subspace of V over F.

Let

$$B_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \middle| a_1, a_2, a_3 \in [0, 24), + \right\} \subseteq V$$

be the special interval vector subspace of V over F.

Clearly B_2 is infinite dimensional over F.

V has several subspaces which are infinite dimensional over F.

Infact if

$$T_1 = \left\{\left.\begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right| a_1 \in [0, 24), +\right\} \subseteq V,$$

$$T_2 = \left\{\left.\begin{bmatrix} 0 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right| a_2 \in [0, 24), +\right\} \subseteq V,$$

$$T_3 = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ a_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right| a_3 \in [0, 24), +\right\} \subseteq V,$$

$$T_4 = \left\{ \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ a_4 \\ 0 \\ 0 \\ 0 \end{array} \right] \middle| \; a_4 \in [0, 24), + \right\} \subseteq V,$$

$$T_5 = \left\{ \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ a_5 \\ 0 \\ 0 \end{array} \right] \middle| \; a_5 \in [0, 24), + \right\} \subseteq V,$$

$$T_6 = \left\{ \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_6 \\ 0 \end{array} \right] \middle| \; a_6 \in [0, 24), + \right\} \subseteq V \text{ and}$$

$$T_7 = \left\{ \left. \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_7 \end{bmatrix} \right| a_7 \in [0, 24), + \right\} \subseteq V$$

be seven distinct special interval vector subspaces of V over F.

Clearly

$$T_i \cap T_j = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}, i \neq j, 1 \leq i, j \leq 7 \text{ and}$$

$T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 = V$ is the distinct sum of subspaces of V over F.

Each subspace is infinite dimensional over F.

Let

$$W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} \middle| \; a_i \in \{0, 8, 16\} \subseteq Z_{24}, 1 \le i \le 7 \right\} \subseteq V$$

be a special interval vector subspace of V of dimension seven over F.

$$\text{Let } S_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_1 \in \{0, 8, 16\} \subseteq Z_{24}, +\} \right. \subseteq V,$$

$$S_2 = \left\{ \begin{bmatrix} 0 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_2 \in \{0, 8, 16\} \subseteq Z_{24}, +\} \right. \subseteq V,$$

$$S_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ a_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_3 \in \{0, 8, 16\} \subseteq Z_{24}, + \right\} \subseteq V,$$

$$S_4 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_4 \in \{0, 8, 16\} \subseteq Z_{24}, + \right\} \subseteq V,$$

$$S_5 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_5 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_5 \in \{0, 8, 16\} \subseteq Z_{24}, + \right\} \subseteq V,$$

$$S_6 = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_6 \\ 0 \end{bmatrix}\right| a_6 \in \{0, 8, 16\} \subseteq Z_{24}, +\right\} \subseteq V \text{ and}$$

$$S_7 = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_7 \end{bmatrix}\right| a_7 \in \{0, 8, 16\} \subseteq Z_{24}, +\right\} \subseteq V$$

be seven different subspaces of V over F.

Each of them is also a subspace of $W \subseteq V$ over F.

Clearly

$$S_i \cap S_j = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}, i \neq j , 1 \leq i, j \leq 7 \text{ and}$$

$S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + S_7 \subseteq V$ is not a direct sum of V but is certainly a direct sum of W.

Thus we call such sum as subdirect sum of subspaces of W.

Infact W can be written as direct sum in other ways also.

Let

$$P_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| a_1, a_2 \in \{0, 8, 16\} \subseteq Z_{24}, + \right\} \subseteq W \subseteq V,$$

$$P_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| a_1, a_2 \in \{0, 8, 16\} \subseteq Z_{24}, + \right\} \subseteq V \text{ and}$$

$$P_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \middle| a_1, a_2, a_3 \in \{0, 8, 16\} \subseteq Z_{24}, + \right\} \subseteq W \subseteq V$$

be special interval vector subspace of W as well as V.

We see

$$P_i \cap P_j = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}, i \neq j, 1 \leq i, j \leq 3$$

and $W = P_1 + P_2 + P_3 \subseteq V$; so P_i's give a subsubdirect of subsubspaces of V.

Infact the representation of V (or $W \subseteq V$) as a direct sum of sub subdirect subsubspace sum is not unique as in case of usual spaces.

Example 2.24: Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| a_i \in [0, 23), 1 \leq i \leq 6, + \right\}$$

be the special interval column matrix vector space over the field Z_{23}.

V has both finite and infinite subspaces. Infact we can have subspaces W in V such that there exists $W^{\perp}$ in V such that $W \oplus W^{\perp} = V$.

Also we have subspaces T in V such that $T^\perp$ of T is only a proper subset of V.

First we will illustrate both these in this V.

$$W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \ a_i \in [0, 23), 1 \leq i \leq 3, + \right\} \subseteq V$$

is a subspace of V over Z_{23}.

We see

$$W^\perp = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \middle| \ a_i \in [0, 23), 1 \leq i \leq 3, + \right\} \subseteq V$$

is a subspace of V and

$$W \cap W^\perp = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ and } W + W^\perp = V.$$

Now take

$$S_1 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_1 \end{bmatrix} \middle| \; a_1 \in [0, 23), + \right\} \subseteq V$$

is a subspace of V and

$$S_1 \cap W = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ but } S + W \neq V$$

we see S is orthogonal with W but $S_1 + W \neq V$.

Similarly

$$S_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ 0 \end{bmatrix} \middle| \; a_1 \in [0, 23), + \right\} \subseteq V$$

is orthogonal with W but $S_1 + W \neq V$ and

$$S_2 \cap W = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Now consider

$$P_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ a_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_1, a_2 \in \{0, 1, 2, \ldots, 22\} \subseteq [0, 23), + \right\}$$

a vector subspace of V. Clearly P_1 is finite dimensional subspace of V over Z_{23}.

$$B_1 = \left\{ \begin{bmatrix} 0 \\ a_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_1 \in [0, 23), + \right\} \subseteq V$$

is such that

$$P_1 \cap B_1 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ and } P_1 + B_1 \neq V$$

infact P_1 is finite dimensional over Z_{23}.

$$B_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \middle| \; a_1 \in [0, 23),\ a_2, a_3 \in Z_{23},\ + \right\} \subseteq V$$

is again a subspace of V of infinite dimensional over $F = Z_{23}$.

$$\text{Clearly } B_2 \cap P_1 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ and } B_2 + P_1 \subseteq V.$$

Thus V has both finite dimensional and infinite dimensional vector subspaces over the field $F = Z_{23}$.

Example 2.25: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in [0, 43), 1 \le i \le 9, + \right\}$$

be a special interval vector space over the field F = Z_{43}. V has several subspaces both of finite and infinite dimension. However V is infinite dimensional over the field F = Z_{43}.

Let

$$W_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1 \in [0, 43), \; + \right\} \subseteq V,$$

$$W_2 = \left\{ \begin{bmatrix} 0 & a_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_2 \in [0, 43), \; + \right\} \subseteq V,$$

$$W_3 = \left\{ \begin{bmatrix} 0 & 0 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_3 \in [0, 43), \; + \right\} \subseteq V,$$

$$W_4 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ a_4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_4 \in [0, 43), \; + \right\} \subseteq V,$$

$$W_5 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_5 \in [0, 43), \; + \right\} \subseteq V,$$

$$W_6 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a_6 \\ 0 & 0 & 0 \end{bmatrix}\right| a_6 \in [0, 43), +\right\} \subseteq V,$$

$$W_7 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_7 & 0 & 0 \end{bmatrix}\right| a_7 \in [0, 43), +\right\} \subseteq V,$$

$$W_8 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_8 & 0 \end{bmatrix}\right| a_8 \in [0, 43), +\right\} \subseteq V \text{ and}$$

$$W_9 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_9 \end{bmatrix}\right| a_9 \in [0, 43), +\right\} \subseteq V$$

are all vector subspaces of V of infinite dimension over the field $F = Z_{43}$.

We see

$$W_i \cap W_j = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\} \text{ if } i \neq j,\ 1 \leq i, j \leq 9$$

and $W_1 + W_2 + \ldots + W_9 = V$ is a direct sum.

This is the maximum number of subspaces of V in which V is written as direct sums can have the number of subspaces to be strictly less than or equal to nine.

Consider

$$R_1 = \left\{ \begin{bmatrix} a_1 & a_2 & 0 \\ 0 & 0 & 0 \\ a_3 & a_4 & 0 \end{bmatrix} \middle| \; a_i \in [0, 43), 1 \leq i \leq 4, + \right\} \subseteq V,$$

$$R_2 = \left\{ \begin{bmatrix} 0 & 0 & a_1 \\ a_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1, a_2 \in [0, 43), 1 \leq i \leq 2, + \right\} \subseteq V \text{ and}$$

$$R_3 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_1 & a_2 \\ 0 & 0 & a_3 \end{bmatrix} \middle| \; a_1, a_2, a_3 \in [0, 43), \; + \right\} \subseteq V$$

are vector subspaces of V over the field $F = Z_{43}$.

$$R_i \cap R_j = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \text{ if } i \neq j, 1 \leq i, j \leq 3 \text{ and}$$

$R_1 + R_2 + R_3 = V$ is thus a direct sum. Each R_i is an infinite dimensional vector subspace of V over F.

Let

$$T_1 = \left\{ \begin{bmatrix} a_1 & a_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \middle| \; a_1, a_2, a_3 \in Z_{43}, \; + \right\} \subseteq V,$$

$$T_2 = \left\{ \begin{bmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ 0 & 0 & a_3 \end{bmatrix} \middle| \; a_1, a_2, a_3 \in Z_{43}, \; + \right\} \subseteq V,$$

$$T_3 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ a_1 & a_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \, a_1, a_2 \in Z_{43}, +\right\} \subseteq V \text{ and}$$

$$T_4 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 & a_2 & 0 \end{bmatrix} \middle| \, a_1, a_2 \in Z_{43}, +\right\} \subseteq V$$

are subspaces of V over the field $F = Z_{43}$.

All the four spaces are finite dimensional over $F = Z_{43}$ and

$$T_i \cap T_j = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \text{ if } i \neq j \text{ and } 1 \leq i, j \leq 4.$$

Further $T_1 + T_2 + T_3 + T_4 \subseteq V$.

Suppose

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \, a_i \in Z_{43}, 1 \leq i \leq 9\right\} \subseteq V,$$

then M is a finite dimensional vector subspace of V over Z_{43}.

Further $M = T_1 + T_2 + T_3 + T_4$ and this sum we call as sub subdirect sum of subsubspaces of the subspace M of V.

The basis for the space

$$M = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \right| a_i \in Z_{43}, 1 \leq i \leq 9 \right\} \text{ is}$$

$$B = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \right.$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\left. \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \subseteq M$$

is a basis of M over Z_{43} and dimension is 9.

Let

$$D = \left\{ \left. \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \right| a_i \in [0, 43); 1 \leq i \leq 3 \right\} \subseteq V$$

be a vector subspace of V over F of infinite dimension

$$E = \left\{ \left. \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \right| a_1, a_2, a_3 \in Z_{43}; + \right\} \subseteq V$$

is a vector subspace of dimension 3 over Z_{43}.

Clearly E $\subseteq$ D; thus a subspace may contain a subspace of finite dimension.

Example 2.26: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \middle| \, a_i \in [0, 47), 1 \leq i \leq 30, + \right\}$$

be the special interval vector space over the field F = Z_{47}.

V has subspaces of finite and infinite dimension over F.

V can be written as a direct sum of subspaces.

If $V = W_1 + \ldots + W_n$; we see n = 30 is the maximum value for n.

Further

$$W_i \cap W_j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \text{ if } i \neq j, 1 \leq i, j \leq n.$$

Just n varies between 2 and 30 that is $2 \leq n \leq 30$.

We have several subspaces of V which in general may not lead to a direct sum.

Let

$$P_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ \vdots & \vdots & \vdots \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_i \in [0, 47), 1 \leq i \leq 18, + \right\} \subseteq V;$$

P_1 is a special interval vector subspace of V over the field $F = Z_{47}$.

$$P_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in [0, 47), 1 \leq i \leq 12, + \right\} \subseteq V;$$

P_2 is a special interval vector subspace of V over the field $F = Z_{47}$.

Clearly

$$P_1 \cap P_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \right\} \text{ and } V = P_1 + P_2;$$

thus V is the direct sum of P_1 and P_2 we see P_1 and P_2 are of infinite dimension over $F = Z_{47}$.

Let

$$S_1 = \left\{ \left. \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \right| a_1 \in [0, 47), + \right\} \subseteq V$$

be the special interval vector subspace of V over F of dimension infinity.

$$S_2 = \left\{ \left. \begin{bmatrix} 0 & a_2 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \right| a_2 \in [0, 47), + \right\} \subseteq V,$$

$$S_3 = \left\{ \left. \begin{bmatrix} 0 & 0 & a_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \right| a_3 \in [0, 47), + \right\} \subseteq V,$$

$$S_4 = \left\{ \left. \begin{bmatrix} 0 & 0 & 0 \\ a_4 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \right| a_4 \in [0, 47), + \right\} \subseteq V \text{ and so on.}$$

$$S_{12} = \left\{ \left. \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & a_{12} \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \right| a_{12} \in [0, 47), + \right\} \subseteq V \text{ and so on.}$$

$$S_{27} = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & a_{27} \\ 0 & 0 & 0 \end{bmatrix}\right| a_{27} \in [0, 47), +\right\} \subseteq V,$$

$$S_{28} = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ a_{28} & 0 & 0 \end{bmatrix}\right| a_{27} \in [0, 47), +\right\} \subseteq V,$$

$$S_{29} = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & a_{29} & 0 \end{bmatrix}\right| a_{29} \in [0, 47), +\right\} \subseteq V \text{ and}$$

$$S_{30} = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & a_{30} \end{bmatrix}\right| a_{30} \in [0, 47), +\right\} \subseteq V$$

are 30 subspaces of V over F which are distinct and each of them are of infinite dimension.

We see

$$P_i \cap P_j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \text{ if } i \neq j,\ 1 \leq i, j \leq 30$$

and $V = P_1 + \ldots + P_{30}$ is a direct sum of subspaces of V.

Example 2.27 : Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \\ a_{31} & a_{32} & \ldots & a_{40} \end{bmatrix} \middle| \; a_i \in [0, 12), 1 \leq i \leq 40, + \right\}$$

be the special interval vector space over $F = \{0, 4, 8\} \subseteq Z_{12}$.

V has several subspaces some of which are infinite dimension and some are of finite dimension.

We can write V as a direct sum of subspaces of V over F.

Thus in view of all these we have the following theorem.

THEOREM 2.4: *Let $V = \{m \times n$ matrices with entries from $[0, n), +\}$ be a special interval vector space over a field $F = Z_n$.*

(1) V has infinite dimensional and finite dimensional subspaces over F.

(2) $V = W_1 + \ldots + W_t$ and $W_i \cap W_j = (0)$, zero matrix if $i \neq j$, $1 \leq i, j \leq t$; with $2 \leq t \leq mn$ where W_i's are vector subspaces of V over F of infinite dimension over F.

Proof is direct and hence left as an exercise to the reader.

Now we proceed onto define the notion of linear algebra using the special interval [0, n) over a field $F \subseteq Z_n$.

DEFINITION 2.2 : *Let $V = \{[0, n), +\}$ be a special interval vector space over a field $F \subseteq Z_n$.*

If on V we define '$\times$' such that $(V, \times)$ is a semigroup, then we define V to be a pseudo special interval linear algebra over F.

We will illustrate this situation by some examples.

Example 2.28: Let V = {[0, 19), +, ×} is a pseudo special interval linear algebra over the field Z_{19}.

We see if x = 16 and y = 10 ∈ V then
x × y = 16 × 10 = 160 = 8 (mod 19) ∈ V.

If x = 0.784 and y = 16 ∈ V then
x × y = 0.784 × 16 = 12.544 ∈ V.

Let x = 5.02 and y = 18 ∈ V

x × y = 5.02 × 18 = 90.36 (mod 19) = 14.36 ∈ V.

We see V is also a linear algebra of infinite dimension over F.

Example 2.29: Let V = {[0, 29), +, ×} be the pseudo special interval linear algebra over the field F = {[0, 4, 8} ⊆ Z_{12}.

W = {{0, 1, 2, 3, …, 11}, +, ×} ⊆ V is a finite dimensional sublinear algebra of V over F.

V has finite dimensional vector subspace which are also linear algebras over F.

However V has vector subspace of finite dimension which are not linear algebras over F.

For take P = {{0, 0.5, 1, 1.5, 2, 2.5, …, 10, 10.5, 11, 11.5}, +} ⊆ V, P is only a vector subspace of V over F.

Clearly P is not a pseudo special sublinear algebra of V over F as 0.5 ×1.5 = 0.75 ∉ P for 0.5 and 1.5 ∈ P hence the claim.

Infact we have several such vector subspaces in V which are not pseudo linear subalgerbas of V over F.

We call these vector subspaces of this special pseudo interval linear algebra as quasi vector subspaces of V.

Example 2.30: Let V = $\{[0, 23), +, \times\}$ be a special pseudo interval linear algebra over the field F = Z_{23}. Take M = $\{\{0, 1, 2, 3, \ldots, 22\}, +, \times\} \subseteq V$ is a sublinear algebra of V over F.

P = $\{\{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 20.5, 21, 21.5, 22, 22.5\}, +, \times\} \subseteq V$ is a special quasi vector subspace of V over F; however P is not closed under product for $1.5 \in P$ but $(1.5)^2 = 2.25 \notin P$ hence P is a special pseudo semilinear subalgebra of V only a vector subspace of V.

Let W = $\{\{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6, \ldots, 22, 22.2, 22.4, 22.6, 22.8\}, +\} \subseteq V$ be a vector quasi subspace of V and is not a pseudo linear subalgebra of V.

Thus V has several vector quasi subspaces which are not linear subalgebras of V.

Example 2.31: Let V = $\{[0, 7), +, \times\}$ be a special pseudo interval linear algebra over a field F = Z_7.

Let $M_1 = \{0, 1, 2, \ldots, 6\} \subseteq V$. M_1 is a special pseudo interval linear subalgebra of V over F.

$M_2 = \{\{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 6, 6.5\}, +\} \subseteq V$ is only a special quasi vector subspace of the linear algebra V over F.

Clearly if $x, y \in M_2$, in general $x \times y \notin M_2$, for take x = 1.5 and $y = 0.5 \in M_2$. $x \times y = 1.5 \times 0.5 = 0.75 \notin M_2$, so M_2 is not a linear subalgebra of V over F.

We have several such special quasi vector subspaces of V which are not pseudo linear subalgebras of the linear algebra V.

Example 2.32: Let V = $\{[0, 2), \times, +\}$ be the special pseudo interval linear algebra over the field F = Z_2.

$P_1 = \{\{0, 0.1, 0.2, \ldots, 0.9, 1, 1.1, 1.2, \ldots, 1.9\} \subseteq [0, 2), +\} \subseteq V$ is a special pseudo interval quasi vector subspace of V over Z_2. Clearly P_1 is not a linear subalgebras of V.

Example 2.33: Let
$V = \{(a_1, a_2, a_3) \mid a_i \in [0, 17), 1 \leq i \leq 3, +, \times\}$ be the pseudo special interval linear algebra over the field $F = Z_{17}$.

We have several sublinear algebras as well as quasi vector subspaces of V over the field F.

We see $T = \{(a_1, a_2, 0) \mid a_1 \in [0, 17), a_2 \in \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 16, 16.5\}, +\}$ is an infinite dimensional quasi vector subspace of V and is not a special linear subalgebra of V.

Let $S = \{(a_1, a_2, a_3) \mid a_i \in \{0, 1, 2, \ldots, 16\} = Z_{17} \subseteq [0, 17)$; $1 \leq i \leq 3, +, \times\} \subseteq V$ is a linear subalgebra of V over $F = Z_{17}$. Clearly S is finite dimension and basis of S is $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. Thus dimension of S over F is 3.

Let $W = \{(0, a_1, a_2) \mid a_1, a_2 \in [0, 17), +, \times\}$ is a linear subalgebra of V over F and dimension of W over F and dimension of W over F is infinite.

$L = \{(a_1, a_2, a_3) \mid a_1, a_2, a_3 \in \{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, \ldots, 16, 16.2, 16.4, 16.6, 16.8\} \subseteq [0, 17), +\}$ is a subvector quasi space of V over F.

Clearly L is not closed under ×, so L is only a quasi vector subspace of V over F so is not a linear subalgebra of V.

For if $x = (0.8, 0.4, 0.6) \in L$ then $x + x = (0.8, 0.4, 0.6) + (0.8, 0.4, 0.6) = (1.6, 0.8, 1.2) \in L$ but $x \times x = (0.8, 0.4, 0.6) \times (0.8, 0.4, 0.6) = (0.64, 0.16, 0.36) \notin L$.

So L is only a quasi vector subspace of V over F.

We see V has several quasi vector subspaces of F which are not linear subalgebras of V over F.

However V can be written as a direct sum of sublinear algebras of V over F.

Let $B_1 = \{(a_1, 0, 0) \mid a_1 \in [0, 17), +, \times\} \subseteq V$,
$B_2 = \{(0, a_2, 0) \mid a_2 \in [0, 17), +, \times\} \subseteq V$ and
$B_3 = \{(0, 0, a_3) \mid a_3 \in [0, 17), +, \times\} \subseteq V$
are all sublinear algebras of V over F of infinite dimension.

Further $B_i \cap B_j = \{(0, 0, 0)\}$ if $i \neq j$, $1 \leq i, j \leq 3$ and $B_1 + B_2 + B_3 = V$.

Thus V is the direct sum of sublinear algebras over F.

Let

$C_1 = \{(a_1, 0, 0) \mid a_1 \in \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 15, 15.5, 16, 16.5\} \subseteq [0, 17), +\} \subseteq V$,

$C_2 = \{(0, a_2, 0, 0) \mid a_2 \in \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 15, 15.5, 16, 16.5\} \subseteq [0, 17), +\} \subseteq V$ and

$C_3 = \{(0, 0, a_3) \mid a_3 \in \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 15, 15.5, 16, 16.5\} \subseteq [0, 17), +\} \subseteq V$

be the quasi vector subspaces of V over $F = Z_{17}$ as C_i is not closed under product so C_i's are not sublinear algebras of V over F.

We have $C_i \cap C_j = \{(0, 0, 0)\}$ if $i \neq j$, $1 \leq i, j \leq 3$ yet $C_1 + C_2 + C_3 = C \subseteq V$ is only a quasi subvector space of V over F and is not a direct sum of V. Infact C is also finite dimensional over F.

The special pseudo interval linear algebra has also sublinear algebras such that the sum is not distinct.

For let

$T_1 = \{(a_1, a_2, 0) \mid a_1, a_2 \in [0, 17), +, \times\} \subseteq V$,
$T_2 = \{(0, a_1, a_2) \mid a_1, a_2 \in [0, 17), +, \times\} \subseteq V$ and
$T_3 = \{(a_1, 0, a_2) \mid a_1, a_2 \in [0, 17), +, \times\} \subseteq V$

be special interval linear subalgebras of V over F. We see $T_i \cap T_j \neq \{(0, 0, 0)\}$ even if $i \neq j$, $1 \leq i, j \leq 3$.

Further $V \subseteq T_1 + T_2 + T_3$ so V is not a direct sum.

Example 2.34: Let $V = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in [0, 38), 1 \leq i \leq 5, +, \times\}$ be the special pseudo interval linear algebra over the field $F = \{0, 19\} \subseteq Z_{38}$.

Clearly V is an infinite dimensional linear algebra over F.

Let $W = \{(a_1, 0, a_2, 0, 0) \mid a_1, a_2 \in [0, 38), +, \times\} \subseteq V$; W is a sublinear algebra of V over F.

$T = \{(0, a_1, 0, 0, 0) \mid a_1 \in \{0, 19\}, +, \times\}$ is a sublinear algebra of V of finite dimension over F.

$W \cap T = \{(0, 0, 0, 0, 0)\}$. We see for every $w \in W$ and for every $t \in T$ we have $t \times w = (0, 0, 0, 0, 0, 0)$.

However $W + T \subseteq V$ and is again an infinite dimensional sublinear algebra of V over F.

Let $N = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in \{0, 0.5, 1, 1.5, 2, \ldots, 18, 18.5, \ldots, 37, 37.5\} \subseteq Z_{19}, 1 \leq i \leq 15, +\} \subseteq V$ is only a finite dimensional quasi vector subspace of V over F and is not a linear subalgebra of V over F.

Example 2.35: Let

$$V = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} \right| a_i \in [0, 6), 1 \leq i \leq 7, +, \times_n \right\}$$

be a special pseudo interval linear algebra of V over the field F = {0, 3} ⊆ Z_6. (Clearly F ≅ Z_2). V is infinite dimensional over F.

Take

$$M = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_7 \end{bmatrix} \right| a_i \in Z_6, 1 \leq i \leq 7, +, \times \right\}$$

is a pseudo linear subalgebra of V over F = {0, 3}.

Clearly M is finite dimensional.

$$\text{Let } x = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 4 \\ 5 \\ 1 \\ 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 5 \\ 2 \\ 1 \\ 0 \\ 4 \\ 3 \\ 2 \end{bmatrix} \in M.$$

$$x \times_n y = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 4 \\ 5 \\ 1 \\ 0 \end{bmatrix} \times_n \begin{bmatrix} 5 \\ 2 \\ 1 \\ 0 \\ 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 3 \\ 0 \\ 2 \\ 3 \\ 0 \end{bmatrix} \in M.$$

$$x + y = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 4 \\ 5 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \\ 1 \\ 0 \\ 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 4 \\ 4 \\ 3 \\ 4 \\ 2 \end{bmatrix} \in M.$$

M is finite dimensional over F.

$$\text{The basis B of M is } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The dimension of M over F is 7.

Let

$$N = \left\{ \begin{bmatrix} a_1 \\ 0 \\ a_2 \\ 0 \\ a_3 \\ 0 \\ a_4 \end{bmatrix} \middle| a_i \in Z_6, 1 \leq i \leq 4, +, \times_n \right\} \subseteq V$$

be a special pseudo linear subalgebra of V over F. Clearly N is finite dimensional sublinear algebra of V.

$$\text{The basis of N is } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Thus dimension of N over F is 4.

Let

$$B = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_7 \end{bmatrix} \middle| a_i \in \{0, 2, 4\} \subseteq Z_6, 1 \leq i \leq 7, +, \times_n \right\} \subseteq V$$

be a special pseudo interval linear subalgebra of V over F.

B is a finite dimensional linear algebra over F.

A basis for B is
$$\left\{\begin{bmatrix}2\\0\\0\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\2\\0\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\2\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\2\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\\2\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\\0\\2\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\\0\\0\\2\end{bmatrix}\right\}.$$

Thus dimension of B over F is 7. Further $|B| < \infty$.

We can write V as a direct sum of sublinear algebras.

If $V = W_1 + \ldots + W_t$, t can maximum be seven and minimum value for t is 2.

Let

$$W_1 = \left\{\begin{bmatrix}a_1\\a_2\\0\\\vdots\\0\end{bmatrix} \middle| a_1, a_2 \in [0, 6), +, \times_n\right\} \subseteq V$$

be a pseudo linear subalgebra of V over F.

$$W_2 = \left\{ \left. \begin{bmatrix} 0 \\ 0 \\ a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right| a_1, a_2 \in [0, 6), +, \times_n \right\} \subseteq V$$

be a linear pseudo subalgebra of V over F.

$$W_3 = \left\{ \left. \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ 0 \\ 0 \end{bmatrix} \right| a_1 \in [0, 6), +, \times_n \right\} \subseteq V$$

be a linear pseudo subalgebra of V over F and

$$W_4 = \left\{ \left. \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \end{bmatrix} \right| a_1, a_2 \in [0, 6), +, \times_n \right\} \subseteq V$$

be a linear pseudo subalgebra of V over F.

Clearly

$$W_i \cap W_j = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ if } i \neq j,\ 1 \leq i, j \leq 4.$$

Further V = $W_1 + W_2 + W_3 + W_4$; thus V is the direct sum of sublinear pseudo algebras over F.

Let

$$L_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| a_1, a_2, a_3 \in [0, 6), +, \times_n \right\} \subseteq V,$$

$$L_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_2 \\ a_1 \\ 0 \\ 0 \end{bmatrix} \middle| a_1, a_2 \in [0, 6), +, \times_n \right\} \subseteq V \text{ and}$$

$$L_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \end{bmatrix} \middle| \; a_1, a_2 \in [0, 6), +, \times_n \right\} \subseteq V$$

be three linear pseudo subalgebras of V over F.

Clearly

$$L_i \cap L_j = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ if } i \neq j, 1 \leq j, i \leq 3 \text{ and}$$

$V = L_1 + L_2 + L_3$ thus V is the direct sum of pseudo sublinear algebras of V over F.

Let

$$B_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \; a_1 \in [0, 6), a_2 \in Z_6, +, \times_n \right\} \subseteq V$$

be a pseudo sublinear algebra of V over F. Clearly it is impossible to find more pseudo sublinear algebras so that B_1 can be in the direct sum of V.

Infact B_1 is infinite dimensional over F; we cannot find B_i's to make them into a direct sum of V over F.

Example 2.36: Let

$$V = \left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}\right| a_i \in [0, 11), 1 \leq i \leq 5, +, \times_n\right\}$$

be a special pseudo interval linear algebra over the field $Z_{11} = F$.

Clearly V is of infinite dimension. V can be written as a direct sum of sublinear algebras.

V has quasi vector subspaces of finite dimension as well as infinite dimension.

For take

$$T_1 = \left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right| a_1, a_2 \in \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 9, 9.5, 10, 10.5\}\right.$$

$$\subseteq [0, 11), +\} \subseteq V$$

is a quasi subvector space of V over F and dimension of T_1 over F is finite.

But

$$T_2 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_1 \in [0, 11)\ a_2 \in \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 9, 9.5, 10, 10.5\} \subseteq [0, 11), + \right\} \subseteq V$$

be the quasi vector subspace of V as if

$$x = \begin{bmatrix} 0.31 \\ 0.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 10 \\ 1.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in T_2$$

$$x \times_n y = \begin{bmatrix} 0.31 \\ 0.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} \times_n \begin{bmatrix} 10 \\ 1.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.1 \\ 0.75 \\ 0 \\ 0 \\ 0 \end{bmatrix} \notin T_2 \text{ as } 0.75 \in \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 9, 9.5, 10, 10.5, 11, 11.5\} \subseteq [0, 11).$$

Thus T_2 is only a quasi vector subspace and is not a pseudo linear subalgebra over F.

Example 2.37: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \middle| \; a_i \in [0, 12), 1 \leq i \leq 18, +, \times_n \right\}$$

be the special pseudo interval linear algebra over the field F = $\{0, 4\} \subseteq Z_{12}$.

V has quasi vector subspaces which are finite dimensional as well as quasi vector subspaces of infinite dimension.

Take

$$M_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1, a_2, a_3 \in [0, 12), 1 \leq i \leq 18, +, \times_n \right\} \subseteq V$$

is a sublinear algebra of V over F of infinite dimension over F.

$$N_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1, a_2, a_3 \in Z_{12}, +, \times_n \right\} \subseteq V$$

is a sublinear algebra of V over F of finite dimension.

$$N_2 = \left\{ \left[\begin{matrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{matrix}\right] \middle| \; a_1, a_2, a_3 \in \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 9, 9.5, 10, 10.5, 11, 11.5\} \subseteq [0, 12), + \right\} \subseteq V$$

is only a quasi vector subspace of V as, in P_1 we cannot define product for if

$$x = \begin{bmatrix} 0.5 & 0.5 & 1.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 2.5 & 0.5 & 1.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \in P, \text{ then}$$

$$x \times_n y = \begin{bmatrix} 0.5 & 0.5 & 1.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \times_n \begin{bmatrix} 2.5 & 0.5 & 1.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1.25 & 0.25 & 2.25 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \notin P_1.$$

Thus P_1 is only a quasi vector subspace of V over F.

Let

$$R_1 = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \right| a_1 \in [0, 12),\ \ a_2, a_3 \in \{0, 0.5, 1, 1.5,\right.$$

$$\left. 2, 2.5, \ldots, 9, 9.5, 10, 10.5, 11, 11.5\} \subseteq [0, 12)\,, +\right\} \subseteq V$$

be a special quasi vector subspace of V over F.

For if

$$x = \begin{bmatrix} 3.2 & 0.5 & 0.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 5 & 1.5 & 2.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \in R_1 \text{ then}$$

$$x \times_n y = \begin{bmatrix} 3.2 & 0.5 & 0.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \times_n \begin{bmatrix} 5 & 1.5 & 2.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0.75 & 1.25 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \notin R_1.$$

We have infinite dimensional quasi subvector spaces as well as finite dimensional quasi subvector spaces.

We can write V as a direct sum of sublinear algebras.

This way of representation of V as a direct sum is not unique.

For

$$V = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{array} \right] \middle| \; a_i \in [0, 12), 1 \le i \le 6, +, \times_n \right\}$$

$$= W_1 \oplus W_2$$

$$= \left\{ \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ \vdots & \vdots & \vdots \\ a_{10} & a_{11} & a_{12} \end{array} \right] \middle| \; a_i \in [0, 12), 1 \le i \le 12, +, \times_n \right\}$$

is the direct sum as both W_1 and W_2 are special pseudo sublinear algebras of V over F and

$$W_1 \cap W_2 = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{array} \right].$$

Similar we can write V as a direct sum of $W_1 + \ldots + W_t$ pseudo sublinear algebras where $2 \le t \le 18$.

T can take the maximum value of 18 and minimum of 2.

Let

$$A_1 = \left\{\left.\begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}\right| a_1 \in [0, 12), +, \times_n\right\} \subseteq V,$$

$$A_2 = \left\{\left.\begin{bmatrix} 0 & a_2 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}\right| a_2 \in [0, 12), +, \times_n\right\} \subseteq V,$$

$$A_3 = \left\{\left.\begin{bmatrix} 0 & 0 & a_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}\right| a_3 \in [0, 12), +, \times_n\right\} \subseteq V \text{ and}$$

so on and

$$A_{18} = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & a_{18} \end{bmatrix}\right| a_{18} \in [0, 12), +, \times_n\right\}$$

are all special pseudo interval linear subalgebras of V over the field F.

Clearly

$$A_i \cap A_j = \{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}\}, 1 \leq i, j \leq 18,$$

further $V = A_1 + A_2 + \ldots + A_{18}$.

Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \right| a_i \in \{0, 1, 2, \ldots, 10, 11\} = Z_{12},$$

$$1 \leq i \leq 18, +, \times_n\}$$

be a special pseudo interval linear subalgebra of V over F.

Clearly P is finite dimensional over F as a linear pseudo subalgebra of V over F.

$P = P_1 + P_2 + \ldots + P_{18}$.

$$\text{where } P_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \right| a_1 \in Z_{12}, +, \times_n\} \subseteq V,$$

$$P_2 = \left\{ \left. \begin{bmatrix} 0 & a_2 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \right| a_2 \in Z_{12}, +, \times_n \right\} \subseteq P \subseteq V, \ldots,$$

$$P_{15} = \left\{ \left. \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & a_{15} \\ 0 & 0 & 0 \end{bmatrix} \right| a_{15} \in Z_{12}, +, \times_n \right\} \subseteq P \subseteq V,$$

$$P_{16} = \left\{ \left. \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ a_{16} & 0 & 0 \end{bmatrix} \right| a_{16} \in Z_{12}, +, \times_n \right\} \subseteq P \subseteq V,$$

$$P_{17} = \left\{ \left. \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & a_{17} & 0 \end{bmatrix} \right| a_{17} \in Z_{12}, +, \times_n \right\} \subseteq P \subseteq V,$$

$$P_{18} = \left\{ \left. \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & a_{18} \end{bmatrix} \right| a_{18} \in Z_{12}, +, \times_n \right\} \subseteq P \subseteq V$$

are all sublinear subalgebras of the linear pseudo subalgebra P over F.

We see $P = P_1 + P_2 + \ldots + P_{18} \subseteq V$ and

$$P_i \cap P_j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}, i \neq j, 1 \leq i, j \leq 18.$$

P is a subdirect subsum of the sublinear pseudo subalgebras of P (or V) but the direct sum does not give V.

We can write $P = P_1 + \ldots + P_t$ where $2 \leq t \leq 18$ but it is subdirect subsum of sublinear algebras. For P properly contains sublinear algebras of the special interval linear algebra V over F.

We see

$$W_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \middle| a_1 \in \{0, 0.5, 1, 1.5, 2, \ldots, 11, 11.5\} \right.$$

$$\subseteq [0, 12), +, \times_n\} \subseteq V$$

is only a special interval quasi vector subspace of V and is not a sublinear algebra of V as; if

$$A = \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \in W, \text{ we see}$$

$$A \times_n B = \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \times_n \begin{bmatrix} 2.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3.75 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \notin W_1,$$

thus W_1 is only a quasi vector subspace of V.

Infact V has several such quasi vector subspaces some of them are of infinite dimensional and others are finite dimensional.

For take

$$W_2 = \left\{ \left[\begin{bmatrix} a & b & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \right| a \in \{0, 0.5, 1, 1.5, 2, \ldots, 11, 11.5\} \text{ and} \right.$$

$$\left. b \in [0, 12), + \right\} \subseteq V,$$

W_2 is only a special quasi vector subspace of V and it is infinite dimensional over F.

We see W_2 is not a linear subalgebra of V over F as product cannot be defined on W_2.

$$\text{Let } A = \begin{bmatrix} 0.5 & 3.12 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1.5 & 0.12 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \in W_2.$$

$$A \times_n B = \begin{bmatrix} 0.5 & 3.12 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \times_n \begin{bmatrix} 1.5 & 0.12 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.75 & 0.3744 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \notin W_2 \text{ as } 0.75 \notin \{0, 0.5, 1, 1.5,$$

$$2, \ldots, 11, 11.5\} \subseteq [0, 12).$$

Thus W_2 is only a special quasi vector subspace of V and is not a sublinear algebra.

Further W_2 is infinite dimensional over F.

V has several such special quasi vector subspace which is infinite dimensional and is not a sublinear algebra of V over F.

Even if we do not use the term pseudo it implies the special linear algebras are pseudo special linear algebras from the very context.

Let

$$W_3 = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ a_4 & a_5 & a_6 \end{bmatrix} \right| a_1, a_4, a_6 \in [0, 12), a_2, a_3, a_5 \in \{0, 0.2, \right.$$

$$\left. 0.4, 0.6, 0.8, 1, 1.2, \ldots, 11, 11.2, \ldots, 11.8\} \subseteq [0, 12) \right\} \subseteq V;$$

W_3 is a quasi vector subspace of V over F and is not a sublinear algebra of V over F. Clearly W_3 is also infinite dimensional over F.

Let

$$M_1 = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \right| a_i \in Z_{12};\ 1 \leq i \leq 18, \times_n, + \right\} \subseteq V$$

is a special interval linear subalgebra of V over the field F.

Clearly M is finite dimensional over F.

Let

$$L_1 = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \right| a_i \in \{0, 0.5, 1, 1.5, 2, \ldots, 11, \right.$$

$$\left. 11.5\} \subseteq [0,12), +;\ 1 \leq i \leq 18 \right\} \subseteq V$$

be a special interval quasi vector subspace of V over F and is not a linear subalgebra of V. Further L_1 is finite dimensional over F.

Example 2.38: Let

$$V = \left\{ \left[\begin{array}{cccc} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \\ a_{31} & a_{32} & \dots & a_{40} \end{array} \right] \middle| \; a_i \in [0, 41),\ 1 \leq i \leq 40, +, \times_n \right\} \subseteq V$$

be the special interval linear algebra over the field $F = Z_{41}$. Clearly V is infinite dimensional over F.

Let

$$M_1 = \left\{ \left[\begin{array}{cccc} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \\ a_{31} & a_{32} & \dots & a_{40} \end{array} \right] \middle| \; a_i \in Z_{41},\ 1 \leq i \leq 40, +, \times_n \right\} \subseteq V$$

be a special pseudo interval linear subalgebra of V over F of finite dimension.

This M_1 has several special pseudo interval linear subalgebras of finite dimension over $F = Z_{41}$.

Let

$$N_1 = \left\{ \left[\begin{array}{cccc} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \\ a_{31} & a_{32} & \dots & a_{40} \end{array} \right] \middle| \; a_i \in \{0, 0.5, 1, 1.5, 2, \dots, 11, 11.5\} \right.$$

$$\left. \subseteq [0, 41),\ 1 \leq i \leq 40, + \right\} \subseteq V$$

be a special quasi interval subvector space of V over F. Clearly N_i is not a linear subalgebra.

For if

$$A = \begin{bmatrix} 0.5 & 0.5 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 1.5 & 0 & 0 & \ldots & 0 \\ 0 & 1.5 & 0 & \ldots & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1.5 & 0.5 & 0 & \ldots & 0 \\ 2 & 0 & 0 & \ldots & 0 \\ 2.5 & 0 & 0 & \ldots & 7.5 \\ 0 & 0.5 & 0 & \ldots & 0 \end{bmatrix} \in N_1.$$

Then

$$A \times_n B = \begin{bmatrix} 0.5 & 0.5 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 1.5 & 0 & 0 & \ldots & 0 \\ 0 & 1.5 & 0 & \ldots & 0 \end{bmatrix} \times_n \begin{bmatrix} 1.5 & 0.5 & 0 & \ldots & 0 \\ 2 & 0 & 0 & \ldots & 0 \\ 2.5 & 0 & 0 & \ldots & 7.5 \\ 0 & 0.5 & 0 & \ldots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.75 & 0.25 & 0 & \ldots & 0 \\ 2 & 0 & 0 & \ldots & 0 \\ 3.75 & 0 & 0 & \ldots & 0 \\ 0 & 0.75 & 0 & \ldots & 0 \end{bmatrix} \notin N_1.$$

Thus N_1 is not special pseudo linear subalgebra of V over F $= Z_{41}$ only a special quasi vector subspace of V over F and is of finite dimension over F.

$$L_1 = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{bmatrix} \middle| a_1, a_2 \in \{0, 0.5, 1, 1.5, 2, \ldots, 11, 11.5, \ldots, 39, 39.5, 40, 40.5\} \subseteq Z_{41} \right\} \subseteq V;$$

L_1 is not a sublinear algebra over $F = Z_{41}$ only a quasi vector subspace of V over Z_{41} and is finite dimensional over Z_{41}.

We have several such quasi vector subspaces some of them finite dimensional and some are infinite dimension.

Now all the special interval linear algebras given by us are of infinite dimensional and infact were commutative.

Now we proceed onto give examples non commutative special interval linear algebras.

Example 2.39: Let

$$V = \left\{\left.\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\right| a_1, a_2, a_3, a_4 \in [0, 13), +, \times\right\}$$

be the special pseudo interval linear algebra over the field $F = Z_{13}$. Clearly if $A, B \in V$ we see $A \times B \neq B \times A$ in general.

Thus V is a non commutative linear algebra of infinite dimension over $F = Z_{13}$.

Let

$$P_1 = \left\{\left.\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\right| a_1 \in \{0, 0.5, 1, 1.5, \ldots., 12, 12.5, +\} \subseteq V\right.$$

is a special quasi vector subspace of V over F.

Infact we have for some A, B in P_1; $A \times B \in V$ but is not in P_1 in general.

$$x = \begin{bmatrix} 1.5 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix} \in P_1.$$

$$x \times y = \begin{bmatrix} 1.5 & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.75 & 0 \\ 0 & 0 \end{bmatrix} \notin P_1.$$

Thus P_1 is not a linear subalgebra only a quasi vector subspace of V.

$$\text{Let } A = \begin{bmatrix} 0.2 & 0 \\ 1 & 0.5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0.5 & 6 \\ 0.2 & 1 \end{bmatrix} \in V.$$

$$\text{We find } A \times B = \begin{bmatrix} 0.2 & 0 \\ 1 & 0.5 \end{bmatrix} \times \begin{bmatrix} 0.5 & 6 \\ 0.2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.10 & 1.2 \\ 0.5 + .10 & 6 + 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.10 & 1.2 \\ 0.15 & 6.5 \end{bmatrix} \qquad \ldots \text{ (I)}$$

$$B \times A = \begin{bmatrix} 0.5 & 6 \\ 0.2 & 1 \end{bmatrix} \times \begin{bmatrix} 0.2 & 0 \\ 1 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1 + 6 & 3 \\ 0.04 + 1 & 0.5 \end{bmatrix} = \begin{bmatrix} 6.1 & 3 \\ 1.04 & 0.5 \end{bmatrix} \qquad \ldots \text{ (II)}$$

Clearly I and II are distinct hence V is only a non commutative linear algebra over F.

Example 2.40: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \, a_i \in [0, 22), 1 \leq i \leq 16, +, \times \right\}$$

be the special interval linear algebra over the field F = {0, 11}.

Clearly V is a non commutative linear algebra. This linear algebra is infinite dimensional.

This has such pseudo special linear algebras which are commutative both finite or infinite dimension.

$$W_1 = \left\{ \left. \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right| a_1 \in [0, 22), +, \times \right\}$$

is a sublinear algebra of V which is commutative and is infinite dimensional.

For if

$$M_1 = \left\{ \left. \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right| a_1 \in Z_{22}, +, \times \right\} \subseteq V$$

be a linear subalgebra of V over F.

Clearly M_1 is commutative over F and is finite dimensional over F.

$$N_1 = \left\{ \left. \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{bmatrix} \right| a_i \in [0, 22), 1 \leq i \leq 4, +, \times \right\} \subseteq V$$

be a pseudo linear subalgebra over the field F = {0, 11}.

N_1 is a commutative pseudo linear subalgebra of infinite dimension over F.

$$M_1 = \left\{ \left[\begin{matrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{matrix} \right] \middle| a_i \in Z_{22}, 1 \leq i \leq 4, +, \times \right\} \subseteq V$$

is a linear subalgebra which is commutative and is of finite dimension over F.

$$N_2 = \left\{ \left[\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & 0 & a_6 & 0 \\ 0 & 0 & 0 & 0 \\ a_7 & a_8 & a_9 & a_{10} \end{matrix} \right] \middle| a_i \in [0, 22), 1 \leq i \leq 10, +, \times \right\} \subseteq V$$

is a sublinear algebra which is non commutative and is of infinite dimension over F.

Let

$$T_1 = \left\{ \left[\begin{matrix} a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \right] \middle| a_1 \in \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 21, 21.5\} \right.$$

$$\left. \subseteq [0, 22), +, \times \right\} \subseteq V$$

be a quasi subvector space as if A, B $\in$ T, A × B $\notin$ T_1 is general.

Hence T_1 is commutative and finite dimensional quasi subvector space of V.

Let

$$P_1 = \left\{ \left. \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ a_3 & 0 & 0 & a_7 \\ 0 & 0 & 0 & 0 \\ 0 & a_4 & a_5 & a_6 \end{bmatrix} \right| a_i \in \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, \right.$$

$$21, 21.5\} \subseteq [0, 22),\ 1 \leq i \leq 7, +, \times\} \subseteq V$$

be a quasi vector subspace of V and is not a sublinear algebra for if A, B $\in P_1$ we see $A \times B \notin P_1$. Thus P_1 is a quasi vector subspace of finite dimension over F.

Example 2.41: Let

$$V = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \right| a_i \in [0, 14), 1 \leq i \leq 9, +, \times \right\}$$

be a non commutative special interval linear algebra over F = {0, 7}. V is infinite dimensional.

As linear algebras V has sublinear algebras. V has quasi subvector subspace over F.

Let

$$M_1 = \left\{ \left. \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \right| a_i \in [0, 14), 1 \leq i \leq 9, +, \times \right\} \subseteq V$$

be an infinite dimensional special pseudo interval sublinear algebra over F.

Clearly M_1 is a commutative pseudo linear algebra over F.

$$M_2 = \left\{ \left. \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \right| a_i \in \{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, \ldots, 13, \right.$$

$$13.2, 13.4, 13.6, 13.8\} \subseteq [0, 14), 1 \leq i \leq 3, +, \times\} \subseteq V$$

is only a quasi vector subspace of V but it is not a linear subalgebra of V as if A, B $\in M_2$ we see A × B $\notin M_2$.

However V is a finite dimensional quasi vector subspace of V which is commutative over F.

Let $A = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.8 \end{bmatrix}$ and $B = \begin{bmatrix} 0.8 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 2.4 \end{bmatrix} \in M_2$;

$$A \times B = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.8 \end{bmatrix} \times \begin{bmatrix} 0.8 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 2.4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.16 & 0 & 0 \\ 0 & .72 & 0 \\ 0 & 0 & 1.92 \end{bmatrix} \notin M_2.$$

Hence the claim.

Let

$$A = \left\{ \left. \begin{bmatrix} a_1 & a_2 & 0 \\ 0 & a_3 & 0 \\ a_4 & 0 & a_5 \end{bmatrix} \right| a_4, a_i \in [0, 14)\ a_2, a_3, a_5 \in \{0, 0.5, 1, 1.5, \right.$$

$$2, \ldots, 12, 12.5, 13, 13.5\} \subseteq [0, 14), +, \times\} \subseteq V$$

be the special quasi vector subspace of V and A is not a linear subalgebra.

Certainly A is only a quasi vector subspace of V of infinite dimension over F.

Thus V has both finite and infinite dimensional quasi vector subspaces.

V has both finite and infinite dimensional linear subalgebras over F.

The question of non commutatively does not arise in case of quasi vector subspaces; however in case of linear subalgebras we may have them to be commutative or otherwise.

Example 2.42: Let
$V = \{(a_1 \mid a_2\ a_3\ a_4 \mid a_5) \mid a_i \in [0, 5), 1 \leq i \leq 5, +, \times\}$ be a special interval linear algebra over the field $F = Z_5$.

V has sublinear algebras as well as quasi vector spaces of infinite and finite dimension.

$M_1 = \{(a_1 \mid 0\ 0\ 0 \mid 0) \mid a_1 \in [0, 5), +, \times\} \subseteq V$ is a sublinear algebra of infinite dimension over F.

$N_1 = \{(a_1 \mid 0\ 0\ 0 \mid 0) \mid a_1 \in Z_5, +, \times\} \subseteq V$ is a sublinear algebra of finite dimension over F.

$W_1 = \{(a_1 \mid 0\ 0\ 0 \mid 0) \mid a_1 \in \{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, \ldots, 4.2, 4.4, 4.6, 4.8\} \subseteq [0, 5), +, \times\} \subseteq V$ is only a quasi vector space we see if

$X = \{(0.2 \mid 0\ 0\ 0 \mid 0)\}$ and $Y = (0.8 \mid 0\ 0\ 0 \mid 0) \in W_1$ then $X \times Y = (0.2 \mid 0\ 0\ 0 \mid 0) \times (0.8 \mid 0\ 0\ 0 \mid 0) = (0.16 \mid 0\ 0\ 0 \mid 0) \notin W_1$, so W_1 is only a special quasi vector subspace of finite dimension over F and is not a sublinear algebra over F.

Let
$W_2 = \{(a_1 \mid a_2\ 0\ 0 \mid a_3) \mid a_1\ a_2 \in Z_5$ and $a_3 \in [0, 5), +, \times\} \subseteq V$ be the special interval linear subalgebra of V over F.

Clearly W_2 is of infinite dimensional over F.

Let $L_1 = \{(0 \mid a_1\ a_2\ a_3 \mid 0) \mid a_i \in Z_5, 1 \leq i \leq 3\} \subseteq V$ be linear subalgebra of V over F. Clearly L_1 is of finite dimension over F.

Let $S_1 = \{(0 \mid a_1\ a_2\ a_3 \mid 0) \mid a_i \in \{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, \ldots, 4.2, 4.4, 4.6, 4.8\} \subseteq [0, 5), 1 \leq i \leq 3\} \subseteq V$ be a quasi subvector space of V over F and is not a linear algebra for if $X = (0 \mid 0.4, 0.8, 1.2 \mid 0)$ and $Y = (0 \mid 0.2, 0.8, 1.2 \mid 0) \in S_1$, then $X \times Y = (0 \mid 0.4, 0.8, 1.2 \mid 0) \times (0 \mid 0.2, 0.8, 1.2 \mid 0) = (0 \mid 0.08, 0.64, 1.44 \mid 0) \notin S_1$, so S_1 is only a quasi vector subspace of V and is not a linear subalgebra of V over F.

Example 2.43: Let

$$V = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \hline a_4 \\ a_5 \\ \hline a_6 \end{bmatrix} \right| a_i \in [0, 12), 1 \leq i \leq 6, +, \times_n \right\}$$

be the special interval linear algebra over $F = \{0, 4, 8\} \subseteq Z_{12}$. Clearly V is commutative and is infinite dimensional over F.

V has sublinear algebras of finite as well as infinite dimension. V also has quasi vector subspaces of both finite and infinite dimension over F.

Let

$$M_1 = \left\{ \left[\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ \hline 0 \\ 0 \\ \hline 0 \end{array} \right] \middle| \; a_i \in [0, 12), 1 \leq i \leq 3, +, \times_n \right\} \subseteq V$$

is a pseudo special sublinear algebra of V over F.

Clearly dimension of M_1 is of infinite cardinality over F.

$$N_1 = \left\{ \left[\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ \hline 0 \\ 0 \\ \hline 0 \end{array} \right] \middle| \; a_i \in Z_{12}, 1 \leq i \leq 3, +, \times_n \right\} \subseteq V$$

is a pseudo special sublinear algebra of V over F. Clearly dimension of N_1 over F is finite.

$$L_1 = \left\{ \left[\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ \hline 0 \\ 0 \\ \hline 0 \end{array} \right] \middle| \; a_i \in \{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, \ldots, 11.2, \ldots, 11.8\} \subseteq [0, Z_{12}), 1 \leq i \leq 3, +, \right\} \subseteq V$$

is only a quasi vector subspace of V over F.

For if A, B $\in L_1$ in general $A \times_n B \not\in V$.

$$\text{Take } A = \left[\begin{array}{c} 0.4 \\ 0.8 \\ 0.6 \\ \hline 0 \\ 0 \\ \hline 0 \end{array}\right] \text{ and } B = \left[\begin{array}{c} 1.2 \\ 0.6 \\ 0.4 \\ \hline 0 \\ 0 \\ \hline 0 \end{array}\right] \text{ in } L_1.$$

$$A \times_n B = \left[\begin{array}{c} 0.4 \\ 0.8 \\ 0.6 \\ \hline 0 \\ 0 \\ \hline 0 \end{array}\right] \times_n \left[\begin{array}{c} 1.2 \\ 0.6 \\ 0.4 \\ \hline 0 \\ 0 \\ \hline 0 \end{array}\right] = \left[\begin{array}{c} 0.48 \\ 0.48 \\ 0.24 \\ \hline 0 \\ 0 \\ \hline 0 \end{array}\right] \not\in L_1 \text{ as } 0.48, 0.24 \not\in \{0, 0.2,$$

$$0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6, 1.8, 2, \ldots, 11, 11.2, 11.4,$$

$$11.6, 11.8\} \subseteq [0, Z_{12}).$$

$$A = \left\{\left[\begin{array}{c} a_1 \\ 0 \\ 0 \\ \hline a_2 \\ 0 \\ \hline a_3 \end{array}\right] \middle| \; a_1 \in [0, 12), a_2, a_3 \in \{0, 0.5, 1, 1.5, \ldots, 11,\right.$$

$$\left. 11.5\} \subseteq [0, 12), +, \times_n\right\} \subseteq V;$$

A is only a quasi vector subspace of V and is not a pseudo special linear subalgebra of V over F for $\times_n$ is not defined on A.

$$\text{Take } x = \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ \hline 0.5 \\ 0 \\ \hline 1.5 \end{bmatrix} \text{ and } y = \begin{bmatrix} 3 \\ 0 \\ 0 \\ \hline 1.5 \\ 0 \\ \hline 0.5 \end{bmatrix} \in A.$$

$$x \times_n y = \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ \hline 0.5 \\ 0 \\ \hline 1.5 \end{bmatrix} \times_n \begin{bmatrix} 3 \\ 0 \\ 0 \\ \hline 1.5 \\ 0 \\ \hline 0.5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ \hline 0.75 \\ 0 \\ \hline 0.75 \end{bmatrix} \notin A \text{ as } 0.75 \notin \{0, 0.5,$$

$$1, 1.5, \ldots, 11, 11.5\} \subseteq [0,12).$$

Thus A is only quasi vector subspace of V over F.

A is an infinite dimensional quasi vector subspace of V over F.

Hence we have quasi subspaces of finite and infinite dimension and similarly sublinear algebras of infinite and finite dimension.

Example 2.44: Let

$$V = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ \hline a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ \hline a_{10} & \cdots & a_{12} \\ a_{13} & \cdots & \cdots \\ a_{16} & \cdots & \cdots \\ \hline a_{19} & \cdots & \cdots \\ \hline a_{22} & \cdots & \cdots \\ a_{25} & \cdots & \cdots \\ a_{28} & a_{29} & a_{30} \end{array} \right] \middle| \; a_i \in [0, 13), 1 \leq i \leq 30, +, \times_n \right\}$$

be a special pseudo interval linear algebra over the field Z_{13}.

This linear algebra is commutative and is infinite dimensional over $Z_{13} = F$.

V has quasi subspaces of finite as well as infinite dimension over F. Infact V has both sublinear algebras of finite and infinite dimension over F. Take

$$P_1 = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ \hline a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ \hline a_{10} & \cdots & a_{12} \\ a_{13} & \cdots & \cdots \\ a_{16} & \cdots & \cdots \\ \hline a_{19} & \cdots & \cdots \\ \hline a_{22} & \cdots & \cdots \\ a_{25} & \cdots & \cdots \\ a_{28} & a_{29} & a_{30} \end{array} \right] \middle| \; a_i \in Z_{13}, 1 \leq i \leq 30, +, \times_n \right\} \subseteq V;$$

P_1 is a sublinear algebra of finite dimension over F.

Let

$$P_2 = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \middle| \; a_i \in [0, 13),\ 1 \leq i \leq 3, +, \times_n \right\} \subseteq V$$

be a special interval linear subalgebra of V over F.

However P_2 is infinite dimensional sublinear algebra over F.

Let

$$P_3 = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \middle| \; a_1, a_2, a_3 \in \{0, 0.5, 1, 1.5, \ldots, 12, 12.5\} \subseteq [0, 13), +, \times \right\} \subseteq V$$

be a special interval quasi vector subspace of V over $F = Z_{13}$. Clearly P_3 is not sublinear algebra over F. Dimension of P_3 over F is finite dimensional. Let

$$P_4 = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \right| a_1, a_2, a_3 \in \{0, 0.5, 1, 1.5, \ldots, 12, 12.5\} \subseteq [0, 13), +, \times \Bigg\} \subseteq V$$

be a special interval quasi vector subspace of V and is not a sublinear algebra over F.

$$P_5 = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ \hline a_4 & a_5 & a_6 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \right| a_i \in Z_{13}, a_j \in [0, 13), 1 \leq i \leq 3, 1 \leq j \leq 3, +, \times_n \Bigg\} \subseteq V$$

is a special interval linear algebra over the field $F = Z_{13}$ of infinite dimension over F.

Thus we can have several such special interval linear subalgebras both of finite and infinite dimension over F = Z_{13}.

Example 2.45: Let

$$V = \left\{ \left(\begin{array}{c|cc|cccc|ccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{20} \\ a_{21} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{30} \end{array} \right) \middle| \; a_i \in [0, 41), 1 \leq i \leq 30, +, \times_n \right\}$$

be the special interval linear algebra over the field F = Z_{41}.

V has several quasi subvector spaces of finite and infinite dimension over F. V has also special interval linear subalgebras over F. Thus this study leads to several interesting results.

Now we can define linear algebras and vector spaces using polynomials.

We call

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S = [0, n), \right.$$

$n < \infty$ the special interval} to be the polynomial ring over the special interval ring ([0, n), +, ×). We see if p(x), q(x) ∈ S[x] then p(x) × q(x) ∈ S[x].

These special pseudo interval rings {[0, n), +, ×} has been introduced earlier in the book [11]. These rings are infact pseudo interval integral domains if n is a prime; only in case of n a non prime S will have zero divisors.

As $n > 1$, clearly $1 \in [0, n)$ serves as the identity with respect to multiplication. S[x] the special interval pseudo polynomial ring can have zero divisor only if S is built using [0, n), n a non prime.

We will illustrate this situation by some examples.

Example 2.46: Let

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 23), +\right\}$$

be the special interval vector space over the field Z_{23}.

Let

$$P[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in Z_{23}\right\} \subseteq S[x]$$

be a special interval vector subspace of S[x] of infinite dimension. The basis for P[x] is $\{1, x, x^2, \ldots, x^n, \ldots, n \to \infty\}$.

However S[x] is infinite dimensional but has a different set of basis.

Let

$$M[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 22.5\} \subseteq [0, 23)\right\} \subseteq S[x]$$

be an infinite dimensional vector subspace of V over F = Z_{23}.

Clearly the dimension of V = S[x] over F is different from the dimension of $M[x] \subseteq V$ over F.

Let

$$T[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 23)\right\} \subseteq S[x]$$

be a subspace of S[x] which is of infinite dimension over F.

Example 2.47: Let

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 62) \right\}$$

be the special interval polynomial vector space over the field F = {0, 31}. S[x] has several subspaces both of finite and infinite dimension.

Take

$$P[x] = \left\{ \sum_{i=0}^{12} a_i x^i \,\middle|\, a_i \in \{0, 31\}, 0 \leq i \leq 12, +\right\} \subseteq S[x]$$

a subspace of S[x] and is finite dimensional for B = {31, 31x, $32x^2$, …, $31x^{12}$} ⊆ P[x] is a basis of P[x] over F = {0, 31}.

Clearly dimension of P[x] is 13 over F.

We take

$$T[x] = \left\{ \sum_{i=0}^{4} a_i x^i \,\middle|\, a_i \in \{0, 31\}, 0 \leq i \leq 4\right\} \subseteq S[x]$$

a vector subspace of S[x] and T[x] is finite dimensional over F and dimension of T[x] is 5 over F and the basis of T[x] is given by B = {31, 31x, $31x^2$, $31x^3$, $31x^4$}. Thus we have several such subspaces of finite dimension over F.

Let

$$D[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \{0, 2, 4, 6, 8, \ldots, 60\} \subseteq Z_{62}\right\} \subseteq S[x]$$

be a infinite dimensional vector subspace of S[x] over F.

Thus S[x] the special interval polynomial vector space has both finite and infinite dimensional vector subspaces over F.

If in S[x] we can define the notion of product then we define S[x] to be the special interval polynomial pseudo linear algebra over the field F.

To this end we will give some more examples.

Example 2.48: Let

$$S[x] = \left\{ \sum_{i=0}^{21} a_i x^i \,\middle|\, a_i \in [0, 41), 0 \leq i \leq 21 \right\}$$

be the special interval polynomial vector space over $F = Z_{41}$. Clearly S[x] is not a special interval polynomial linear pseudo algebra as × cannot be defined on S[x].

Let $p(x) = x^{20} + 3x^2 + 1$ and $q(x) = 3x^{12} + 8x + 31 \in S[x]$

$$\begin{aligned} p(x) \times q(x) &= (x^{20} + 3x^2 + 1) \times (3x^{12} + 8x + 31) \\ &= 3x^{32} + 9x^{14} + 3x^{12} + 8x^{21} + 8x^2 + 24x^3 + 93x^2 + 31x^{20} + 31 \\ &= 3x^{32} + 8x^{21} + 31x^{20} + 9x^{14} + 3x^{12} + 24x^3 + 19x^2 + 31 \notin S[x]. \end{aligned}$$

This product cannot be defined in S[x], so S[x] is only special interval polynomial vector space and not a special interval pseudo linear algebra.

Example 2.49: Let

$$S[x] = \left\{ \sum_{i=0}^{7} a_i x^i \,\middle|\, a_i \in [0, 12), 0 \leq i \leq 7, + \right\}$$

be a special interval vector space of polynomials over the field $F = \{0, 8, 4\} \subseteq Z_{12}$. S[x] is only a vector space and not a special pseudo linear algebra.

S[x] is infinite dimensional over F. However S[x] has finite dimensional subspaces as well as infinite dimensional subspaces.

Let

$$P[x] = \left\{ \sum_{i=0}^{7} a_i x^i \,\middle|\, a_i \in \{0, 2, 4, 6, 8, 10\} \subseteq Z_{12}, 0 \leq i \leq 7 \right\} \subseteq S[x]$$

be a finite dimensional vector subspace of S[x] over F.

Let

$$T[x] = \left\{ \sum_{i=0}^{7} a_i x^i \,\middle|\, a_i \in \{0, 1, 0.5, 1.5, 2, 2.5, 3, 3.5, \ldots, 11, 11.5\} \subseteq [0, 12), 0 \leq i \leq 7 \right\} \subseteq S[x]$$

be a finite dimensional polynomial vector subspace of S[x] over $F = \{0, 8, 4\} \subseteq Z_{12}$.

Let

$$V[x] = \left\{ \sum_{i=0}^{3} a_i x^i \,\middle|\, a_i \in [0, 12), 0 \leq i \leq 3 \right\} \subseteq S[x]$$

is an infinite dimensional vector subspaces of S[x] over F.

Thus S[x] has both finite and infinite dimensional polynomial vector subspaces. However S[x] is never a pseudo special linear algebra.

Inview of all these we just state the following theorem the proof of which is direct.

THEOREM 2.5: *Let V be a special interval pseudo linear algebra defined over a field F. V is always a special interval vector space over F however every special interval vector space defined over a field F in general is not a special interval pseudo linear algebra.*

Example 2.50: Let

$$V = S[x] = \left\{\sum_{i=0}^{12} a_i x^i \,\middle|\, a_i \in [0, 7), 0 \leq i \leq 12\right\}$$

be the special pseudo interval polynomial vector space over the field Z_7. Clearly V is never a linear algebra.

Dimension of S[x] over F is infinite. However S[x] has subspaces of finite dimension. For take

$$T[x] = \left\{\sum_{i=0}^{12} a_i x^i \,\middle|\, a_i \in Z_7, 0 \leq i \leq 12\right\} \subseteq V$$

is a subspace of V but T[x] is of finite dimension over $F = Z_7$.

Let

$$W[x] = \left\{\sum_{i=0}^{5} a_i x^i \,\middle|\, ai \in \{0, 0.2, 0.4, 0.6, 0.8,, 1, 1.2, \ldots, 6, 6.2, 6.4, 6.6, 6.8\} \subseteq [0, 7), 0 \leq i \leq 5\right\} \subseteq V;$$

W[x] is a finite dimensional vector subspace of S[x] over F.

$$B[x] = \left\{\sum_{i=0}^{5} a_i x^i \,\middle|\, a_i \in [0, 7), 0 \leq i \leq 5\right\} \subseteq S[x]$$

is an infinite dimensional vector subspace of x over the field F.

However we can have special interval polynomial pseudo linear algebras over a field.

We will illustrate this situation by a few examples.

Example 2.51: Let

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 23), +, \times \right\}$$

is a special interval polynomial pseudo linear algebra over the field $F = Z_{23}$.

S[x] has quasi vector subspaces which are not linear subalgebras both of finite and infinite dimension over $F = Z_{23}$.

Also S[x] has linear pseudo subalgebras all of them are infinite dimensional over F.

Let

$$P[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \{0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, \ldots, 22, 22.5\} \subseteq [0, 23) \right\} \subseteq S[x]$$

be a subspace of S[x] over F. Clearly P[x] is an infinite dimensional quasi vector subspace of S[x] over F.

P[x] is not a pseudo linear subalgebra of S[x] over F.

Let

$$T[x] = \left\{ \sum_{i=0}^{20} a_i x^i \,\middle|\, a_i \in Z_{23}, 0 \leq i \leq 20, + \right\} \subseteq S[x]$$

be a finite dimensional quasi subset vector space over F.

Clearly T[x] is not a pseudo sublinear algebra.

Infact

$$B[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \right| a_i \in Z_{23}, +, \times\} \subseteq S[x]$$

is a pseudo linear subalgebra of S[x] over F and dimension of B[x] is infinite over F.

$$D[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \right| a_i \in [0, 23), +, \times\} \subseteq S[x]$$

is a linear pseudo subalgebra of infinite dimension over F.

Clearly D[x] has uncountable infinite basis.

Example 2.52: Let

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \right| a_i \in [0, 34), +, \times\}$$

be a special interval pseudo linear algebra over the field F = {0, 17} ⊆ Z_{34}.

S[x] has quasi vector subspaces also linear pseudo subalgebras of finite and infinite dimension.

$$\text{Let } T[x] = \left\{ \sum_{i=0}^{5} a_i x^i \right| a_i \in \{0, 17\}, 0 \leq I \leq 5, +\} \subseteq S[x]$$

be a special quasi vector subspace of V over F and dimension of T[x] is finite. However T[x] is not closed under product.

$$L[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \right| a_i \in Z_{34}, +, \times\} \subseteq V$$

is a special interval pseudo linear subalgebra of V over F.

Clearly L[x] is infinite dimensional over F.

$$M[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \right| a_i \in \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 33.5\}, +\}$$

is only a quasi vector subspace over F and it is not a pseudo linear subalgebra as if $p(x) = 0.5x^3 + 1.5x^2 + 0.5$ and $q(x) = 1.5x^2 + 0.5x + 0.5 \in M[x]$.

$$\begin{aligned} p(x) \times q(x) &= (0.5x^3 + 1.5x^2 + 0.5) \times (1.5x^2 + 0.5x + 0.5) \\ &= 075x^5 + 2.25x^4 + 0.75x^2 + 0.25x^4 + 0.75x^2 + 0.5 \\ &= 0.75x^5 + 2.5x^4 + 0x^3 + 1.5x^2 + 0.25x + 0.25 \notin M[x]. \end{aligned}$$

Thus M[x] is only a quasi subvector space over F.

Clearly M[x] is an infinite dimensional over F.

$$W[x] = \left\{ \sum_{i=0}^{5} a_i x^i \right| a_i \in [0, 34), 0 \leq i \leq 5, +\} \subseteq V;$$

W[x] is a infinite dimensional vector subspace over F but W[x] is not a linear pseudo subalgebra over F.

$$D[x] = \left\{ \sum_{i=0}^{7} a_i x^i \right| a_i \in \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 33.5\}, 0 \leq i \leq 7, +\} \subseteq S[x]$$

be a special quasi polynomial vector subspace over F.

Clearly D[x] is finite dimensional over F.

Example 2.53: Let

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \right| a_i \in [0, 35), +, \times\}$$

is a special interval pseudo linear algebra over the field $F = \{0, 7, 14, 21, 28\} \subseteq Z_{35}$. S[x] is infinite dimensional over F. S[x] has quasi subspaces both of finite and infinite dimensional.

Let

$$M[x] = \left\{ \sum_{i=0}^{7} a_i x^i \right| a_i \in F, 0 \leq i \leq 7\} \subseteq S[x]$$

be the quasi vector subspace over F. Clearly M[x] is not a linear pseudo subalgebra.

Let

$$W[x] = \left\{ \sum_{i=0}^{5} a_i x^i \right| a_i \in F, 0 \leq i \leq 5\} \subseteq S[x]$$

be the special linear pseudo subalgebra of V over F. Clearly dimension of W[x] is infinite over F.

$$B[x] = \left\{ \sum_{i=0}^{8} a_i x^i \right| a_i \in \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 34.5\}$$

$$\subseteq [0, 35), 0 \leq i \leq 8, +\} \subseteq V;$$

is a special quasi vector subspace of V of finite dimension over F. Clearly B[x] is not a linear pseudo subalgebra of V over F.

Infact V has infinitely many finite dimensional quasi vector subspaces over F. Also V has infinitely many infinite dimensional quasi vector subspaces over F where none of them are linear pseudo subalgebras of V over F.

It is important to note none of the pseudo sublinear algebras are finite dimensional.

We see these polynomial pseudo linear algebra have sublinear pseudo algebras of infinite dimensions where at least one is of countable infinity.

Example 2.54: Let

$$V = S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 43), +, \times \right\}$$

be the special interval pseudo linear algebra over the field $F = Z_{43}$.

Let

$$M_1 = \left\{ \sum_{i=5}^{9} a_i x^i \,\middle|\, a_i \in [0, 43), 5 \leq i \leq 9, +, \times \right\} \subseteq V$$

be a special interval quasi subspace over F and M_1 is not a linear pseudo subalgebra of V over F.

For if $p(x) = x^5 + 10x^8 + 20x^9$
and $q(x) = 6x^7 + 2x^6 \in M_1$

$$\begin{aligned} \text{Now } p(x) \times q(x) &= (x^5 + 10x^8 + 20x^9) \times (6x^7 + 2x^6) \\ &= 6x^{12} + 60x^{15} + 120x^{16} + 40x^{15} \\ &= 6x^{12} + 40x^{15} + 17x^{15} + 34x^{16} \\ &= 6x^{12} + 34x^{16} + 14x^{15} \notin M_1. \end{aligned}$$

As all polynomials in M_1 is of degree less than or equal to 9 and greater than or equal to 5. Thus M_1 can only be a quasi special subvector space of V over F.

$$N_1 = \left\{\sum_{i=0}^{7} a_i x^i \,\middle|\, a_i \in [0, 43), +, 0 \leq i \leq 7\right\} \subseteq V$$

be the special interval quasi vector subspace of V over F. Clearly N_1 is infinite dimensional over F but N_1 is not a pseudo linear subalgebra of V over F.

$$N_2 = \left\{\sum_{i=0}^{9} a_i x^i \,\middle|\, a_i \in Z_{43}, 0 \leq i \leq 9\right\} \subseteq V$$

is a special quasi vector subspace of V over F. Clearly N_2 is finite dimensional.

Further N_2 is not a linear pseudo subalgebra of V over F.

$$N_3 = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \{0, 0.5, 1, 1.5, \ldots, 41, 41.5, 42,\right.$$

$$\left. 42.5\}, \subseteq [0, 43), +, \times\right\} \subseteq V$$

is a special quasi vector subspace of V of infinite dimension over F.

Clearly N_3 is not a linear pseudo subalgebras as $p(x) = 9.5x^5 + 0.5$ and $1.5x^8 + 0.5$ and $q(x) = 1.5x^8 + 0.5 \in N_2$ then $q(x) \times p(x) = (1.5x^8 + 0.5) \times (9.5x^5 + 0.5)$

$= 14.25x^{13} + 4.75x^5 + 0.75x^8 + 0.25 \notin N_3$ as none of the coefficients are in $\{0, 0.5, 1, 1.5, 2, \ldots, 42, 42.5\} \subseteq \{[0, 43)\}$. Hence N_3 is a linear pseudo subalgebra of V over F.

Now we proceed onto suggest some problems some of which are very difficult and can be realized as research problems.

Problems

1. Obtain some special features enjoyed by special interval vector spaces.

2. Distinguish the properties between special interval linear algebras and vector spaces.

3. Give an example of a special interval vector space which is not a special interval linear algebra.

4. Show all special interval linear algebras are infinite dimensional.

5. Let V = {[0, 21), +} be the special interval vector space over the field F = {0, 7, 14} $\subseteq Z_{21}$.

   (i) Show V is infinite dimensional over F.
   (ii) Find 5 subspaces of V of finite dimension over F.
   (iii) Can V have subspaces of infinite dimension over F?
   (iv) Can V be written as a direct sum of subspaces?

6. Let V = {[0, 47), +} be the special interval vector space over the field Z_{47}.

   Study questions (i) to (iv) of problem 5 for this V.

7. Let V = {[0, 29), +} be the special interval vector space over the field Z_{29}.

   Study questions (i) to (iv) of problem 5 for this V.

8. Let V = {[0, 6), +} be the special interval vector space over F = {0, 3} $\subseteq Z_6$.

   (i) Study questions (i) to (iv) of problem 5 for this V.
   (ii) Study questions (i) to (iv) of problem 5 for this V if F is replaced by the field {0, 2, 4} $\subseteq Z_6$.

9. Let $V = \{[0, 2p), p$ a prime, $+\}$ be the special interval vector space over the field $F = \{0, p\} \subseteq [0, 2p)$.

   (i) Study questions (i) to (iv) of problem 5 for this V.
   (ii) Prove Z_{2p} has only two subsets which are fields.

10. Let $V = \{[0, 24), +\}$ be a special interval vector space over a field $F \subseteq Z_{24}$.

    (i) How many subsets in Z_{24} are fields?
    (ii) Study questions (i) to (iv) of problem 5 for this V.

11. Let $V = \{[0, Z_{30}), +\}$ be a special interval vector space over a field $F = \{0, 10, 20\} \subseteq Z_{30}$.

    (i) Study questions (i) to (iv) of problem 5 for this V.
    (ii) Find all subsets in Z_{30} which are fields of Z_{30}.

12. Prove all special interval vector spaces $V = \{[0, n), +, n < \infty\}$ are always closed under × mod n but product does not in general distribute over addition, that is $a \times (b + c) \neq a \times b + a \times c$ for all $a, b, c \in [0, n)$. Hence V is a special interval linear algebra over $F \subseteq Z_n$.

    (i) Prove V has quasi vector subspaces which are not special pseudo linear subalgebras over F.

13. Is it possible to have a special interval pseudo linear algebra which has no linear pseudo subalgebras?

14. Is it possible to have special interval linear pseudo algebra which has no special quasi vector subspaces?

15. Let $V = \{[0, 28), +, \times\}$ be a special interval pseudo linear algebra over a field $F \subseteq Z_{28}$.

    (i) Find special interval linear pseudo subalgebras of V over F.

(ii) Can V have infinite number of linear pseudo subalgebras over F?
(iii) Is it possible to have linear pseudo subalgebras of V of finite dimension over F?
(iv) Can V have special quasi vector subspaces of infinite dimension over F?
(v) Is it possible for V to have special quasi vector subspaces of finite dimension over F?
(vi) How many infinite number of basis can V have?
(vii) Find all subsets in Z_{28} which are subfields of Z_{28}.

16. Let V = $\{[0, 48), +, \times\}$ be a special interval pseudo linear algebra over a field $F \subseteq Z_{48}$.

Study questions (i) to (vii) of problem 15 for this V.

17. Can Z_{p^2}, p a prime be a S- pseudo special interval ring?

18. Let V = $\{[0, 660), \times, +\}$ be a special interval pseudo linear algebra over $F \subseteq Z_{660}$.

Study questions (i) to (vii) of problem 15 for this V.

19. Let V = $\{[0, 420), \times, +\}$ be a special interval pseudo linear algebra over $F \subseteq Z_{420}$.

Study questions (i) to (vii) of problem 15 for this V.

20. Let V = $\{(a_1, a_2, a_3, a_4) \mid a_i \in [0, 83), 1 \leq i \leq 4, +, \times\}$ be the special interval pseudo linear algebra over $F = Z_{83}$.

(i) Find all sublinear pseudo algebras of V over F of finite dimension.
(ii) Find all quasi vector subspaces of V over F of finite and infinite dimension over F.
(iii) Find all linear operators of V.
(iv) Can V be written as a direct sum of sublinear pseudo algebras over F?

(v) Find the algebraic structure enjoyed by
$$V_T = \{T : V \to V\}.$$

21. Let $V = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in [0, 42), 1 \leq i \leq 5, +, \times\}$ the special interval pseudo linear algebra over $F \subseteq Z_{42}$ (F a field in Z_{42}).

(i) Find all subsets of Z_{42} which are fields in Z_{42}.
(ii) Study questions (i) to (vii) of problem 15 for this V over all the fields in Z_{42}.

22. Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_7 \end{bmatrix} \middle| \; a_i \in [0, 19; 1 \leq i \leq 7, +, \times_n \right\}$$

be the special interval pseudo linear algebra over $F = Z_{19}$.

(i) Study questions (i) to (vii) of problem 15 for this V over all the fields in Z_{42}.

23. Let
$V = \{(a_1 \mid a_2\ a_3\ a_4 \mid a_5\ a_6 \mid a_7) \mid a_i \in [0, 46), 1 \leq i \leq 7, +, \times\}$ be the special interval pseudo linear algebra over a field $F \subseteq Z_{46}$.

(i) Study questions (i) to (vii) of problem 15 for this V.
(ii) Find all subspaces (sublinear pseudo algebras) which are orthogonal to $P = \{(a_1 \mid 0\ 0\ 0 \mid a_2\ 0 \mid a_3) \mid a_i \in [0, 46), 1 \leq i \leq 3, +, \times\} \subseteq V$.

24. Let $V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \middle| \; a_i \in [0, 86), \; 1 \leq i \leq 10, +, \times_n \right\}$ be the

special interval pseudo algebra over a field $\{0, 43) \subseteq Z_{86}$.

Study questions (i) to (vii) of problem 15 for this V.

25. Let

$$V = \left\{ \left[\begin{array}{c} a_1 \\ \hline a_2 \\ a_3 \\ a_4 \\ \hline a_5 \\ a_6 \\ \hline a_7 \end{array} \right] \middle| \; a_i \in [0, 22),\; 1 \leq i \leq 7, +, \times_n \right\}$$

be the special pseudo interval algebra over a field $F \subseteq Z_{22}$.

Study questions (i) to (vii) of problem 15 for this V.

26. Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & \ldots & \ldots & \ldots & \ldots & a_{12} \\ a_{13} & \ldots & \ldots & \ldots & \ldots & a_{18} \\ a_{19} & \ldots & \ldots & \ldots & \ldots & a_{24} \\ a_{25} & \ldots & \ldots & \ldots & \ldots & a_{30} \\ a_{31} & \ldots & \ldots & \ldots & \ldots & a_{36} \end{bmatrix} \middle| \; a_i \in [0, 61),\; 1 \leq i \leq 36, +, \times \right\}$$

be the special pseudo interval algebra over a field $F = Z_{61}$. Clearly V is a non commutative pseudo linear algebra over F.

(i) Study questions (i) to (vii) of problem 15 for this V.
(ii) Find all commutative linear pseudo subalgebras and commutative vector subspaces of V over $F = Z_{61}$.

27. Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \; a_i \in [0, 11), 1 \leq i \leq 16, +, \times \right\}$$

be the special interval linear pseudo algebra over $F = Z_{11}$.

(i) Study questions (i) to (vii) of problem 15 for this V.
(ii) When are these sublinear pseudo algebras Commutative?
(iii) Find those subvector spaces which are commutative.

28. Let

$$V = \left\{ \left[\begin{array}{cc|ccc|c} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & \ldots & \ldots & \ldots & \ldots & a_{12} \\ \hline a_{13} & \ldots & \ldots & \ldots & \ldots & a_{18} \\ a_{19} & \ldots & \ldots & \ldots & \ldots & a_{24} \\ a_{25} & \ldots & \ldots & \ldots & \ldots & a_{30} \\ \hline a_{31} & \ldots & \ldots & \ldots & \ldots & a_{36} \end{array} \right] \middle| \; a_i \in [0, 23), \right.$$

$$\left. 1 \leq i \leq 36, +, \times \right\}$$

is a special interval linear pseudo algebra over $F = Z_{23}$. Clearly V is non commutative.

(i) Study questions (i) to (vii) of problem 15 for this V.
(ii) Study questions (ii) of problem 26 for this S.

29. Let

$$V = \left\{\left[\begin{array}{cccc} a_1 & a_2 & \cdots & a_5 \\ a_6 & a_7 & \cdots & a_{10} \\ a_{11} & a_{12} & \cdots & a_{15} \\ a_{16} & a_{17} & \cdots & a_{20} \\ a_{21} & a_{22} & \cdots & a_{25} \end{array}\right] \middle| \; a_i \in [0, 15),\ 1 \leq i \leq 25, +, \times_n\right\}$$

be the special interval pseudo linear algebra over the field $F \subseteq Z_{15}$ ($F \neq \phi$ a subset which is a field).

(i) Study questions (i) to (vii) of problem 15 for this V.

30. Prove all special interval polynomial linear pseudo algebras are always of infinite dimension (if $x^n = 1$ is not used for $n < \infty$) over the field on which it is defined.

31. Obtain some special and interesting features enjoyed by special interval polynomial linear pseudo algebra defined over a field F.

32. Let

$$V = \left\{\left[\begin{array}{cc|ccc|c} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & \cdots & \cdots & \cdots & \cdots & a_{12} \\ \hline a_{13} & \cdots & \cdots & \cdots & \cdots & a_{18} \\ a_{19} & \cdots & \cdots & \cdots & \cdots & a_{24} \\ \hline a_{25} & \cdots & \cdots & \cdots & \cdots & a_{30} \end{array}\right] \middle| \; a_i \in [0, 43),\ 1 \leq i \leq 30, +, \times_n\right\}$$

be the special interval linear pseudo algebra over the field Z_{43}.

Study questions (i) to (vii) of problem 15 for this V.

33. Let

$$V = \left\{ \left(\begin{array}{cc|ccc|c} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \end{array} \right) \middle| \; a_i \in [0, 241), \right.$$
$$\left. 1 \leq i \leq 18, +, \times_n \right\}$$

be the special interval linear pseudo algebra over the field Z_{241}.

Study questions (i) to (vii) of problem 15 for this V.

34. Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \; a_i \in [0, 3), \; 1 \leq i \leq 4, +, \times \right\}$$

be the special interval linear pseudo algebra over the field $F = Z_3$.

Study questions (i) to (vii) of problem 15 for this V.

35. Give special features enjoyed by special pseudo interval linear algebras.

36. Let

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \middle| \; a_j \in [0, 7), +, \times \right\}$$

be the special interval polynomial linear pseudo algebra over the field $Z_7 = F$.

(i) Study the special linear pseudo subalgebras of V over F.

(ii) Will every sublinear pseudo algebra of V over F be infinite dimension?
(iii) Find quasi vector subspaces of V over F of both finite and infinite dimension over F.

37. Let

$$V = \left\{ \sum_{i=0}^{19} a_i x^i \,\middle|\, a_j \in [0, 46), 0 \leq i \leq 19, +\right\}$$

be a special interval vector space over the field $F = \{0, 23\} \subseteq Z_{46}$.

(i) Show V is not a special linear pseudo algebra.
(ii) Prove dimension of V over F is infinite.
(iii) Find vector subspaces of V which are finite dimensional over F.
(iv) How many vector subspaces of V over F are finite dimensional?
(v) Find all vector subspaces of V over F of infinite dimension.
(vi) Can subspaces of V of finite dimension have more than one basis?
(vii) Is it possible to have a vector subspace of dimension 8?

38. Let

$$V = \left\{ \sum_{i=0}^{90} a_i x^i \,\middle|\, a_j \in [0, 5), 0 \leq i \leq 90, +\right\}$$

be a special interval vector space over $F = Z_5$.

Study questions (i) to (vii) of problem 37 for this V.

39. What will happen if [0, 5) in problem 38 is replaced by [0, 51)?
Study questions (i) to (vii) of problem 37 for this V with [0, 5) replaced by [0 51).

Chapter Three

SMARANDACHE SPECIAL INTERVAL PSEUDO LINEAR ALGEBRAS

In this chapter we define two new concepts viz., Smaradache special interval pseudo linear algebra (S-special interval pseudo linear algebra) and Smarandache strong special interval pseudo linear algebra. They are illustrated by examples and described and developed in this chapter.

DEFINITION 3.1: *Let S = {[0, n), +} be the additive abelian group. Let $Z_n \subseteq [0, n)$ be Smarandache ring. Let S be a special interval vector space we define S as a Smarandache special interval vector space over the S-ring Z_n.*

Here instead of the field in Z_n we use the totality of Z_n. We give examples of them.

Example 3.1: Let S = {[0, 6), +} be a S-special interval vector space over the S-ring Z_6.

Example 3.2: Let S = {[0, 15), +} be a S-special interval vector space over a ring S-ring Z_{15}.

Example 3.3: Let S = {[0, 14), +} be a S-special interval vector space over the S-ring Z_{14}.

Example 3.4: Let S = {[0, 46), +} be a S-special interval vector space over the S-ring Z_{46}.

Example 3.5: Let S = {[0, 21), +} be the S-special interval vector space over the S-ring Z_{21}.

We prove the following theorems.

THEOREM 3.1: *Let S = {[0, 2p), +}, p a prime; S is a S-special interval vector space over the S-ring* Z_{2p}.

Proof follows from the simple fact as Z_{2p} is a S-ring for $\{0, p\} \subseteq Z_{2p}$ is a field.

THEOREM 3.2: *S = {[0, 3p), +} (p a prime) is a S-special interval vector space over the S-ring* Z_{3p}.

Proof follows from the fact Z_{3p} is a S-ring. Hence the claim.

THEOREM 3.3: *S = {[0, pq), +} (p and q are primes) is a S-special interval vector space over the S-ring* Z_{pq}.

Example 3.6: Let S = {[0, 33), +} be the S-special interval vector space over the S-ring Z_{33}.

We will describe vector subspaces of S over the S-ring.

Example 3.7: Let S = {[0, 14), +} be a special interval vector space over the S-ring Z_{14}. S is infinite dimensional over Z_{14}.

Let

P = {{0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, …, 11.5, 12, 12.5, 13, 13.5}, +} be a S-special interval vector subspace of S over the S-ring Z_{14}. Clearly P is finite dimensional over the S-ring Z_{14}.

M = {{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, …, 13, 13.2, 13.4, 13.6, 13.8}, +} $\subseteq$ S is again a finite dimensional vector subspace of S over Z_{14}. We have infinite number of finite dimensional vector subspaces over Z_{14}.

T = {{0, 0.1, 0.2, …, 0.9, 1, 1.1, 1.2, …, 2, 2.1, 2.2, …, 13, 13.1, 13.2, …, 13.9}, +} $\subseteq$ S be the finite dimensional vector subspace of V over the S-ring Z_{14}.

Example 3.8: Let S = {[0, 33), +} be a S-special interval vector space of V over the S-ring R = Z_{33}.

M_1 = {{0, 1, 2, …, 32), +} $\subseteq$ V be the S-special vector subspace of V of dimension 1.

M_2 = {{0. 0.5, 1, 1.5, 2, 2.5, …, 31, 31.5, 32, 32.5}, +} $\subseteq$ V be the S-special interval vector subspace of V over the S-ring R of finite dimension.

M_3 = {{0, 0.1, 0.2, …, 0.9, 1, 1.1, 1.2, …, 1.9, 2, 2.1, …, 2.9, …, 32.1, 32.2, …, 32.9}, +} $\subseteq$ V be the S-special interval vector subspace of V over the S-ring R.

Clearly M_3 is finite dimensional vector subspace of V over R.

M_4 = {{0, 0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75, 2, …, 32, 32.25, 32.5, 32.75}, +} $\subseteq$ V be the S-special interval vector subspace of V over the S-ring R.

Thus V has several finite dimensional S-special interval vector subspaces of V over R.

Example 3.9: Let V = {[0, 35), +} be the S-special interval vector space over the S-ring R = Z_{35}.

T_1 = {{0, 1, 2, …, 34}, +} $\subseteq$ V be subvector space of V over R.

$T_2 = \{\{0, 5, 10, 15, 20, 25, 30\}, +\} \subseteq V$ be the vector subspace of V over R.

Both T_1 and T_2 are finite dimensional over R.

$T_3 = \{\{0, 7, 14, 21, 28\}, +\} \subseteq V$ be the vector subspace of V over R. T_3 is also finite dimensional over Z_{35}.

$T_4 = \{\{0, 0.1, 0.2, 0.3, \ldots, 0.9, 1, 1.1, 1.2, \ldots, 2, \ldots, 30, 3.1, 3.2, \ldots, 30.9, \ldots, 34.1, 34.2, \ldots, 34.9\}, +\} \subseteq V$ is also a vector subspace of V over $R = Z_{35}$.

T_4 is also finite dimensional over Z_{35} and so on.

Example 3.10: Let $V = \{[0, 28), +\}$ be the S-special interval vector space over the S-ring $R = Z_{28}$.

$P_1 = \{\{0, 2, 4, 6, 7, \ldots, 26\}, +\}, \subseteq V$ is a S-vector subspace of V of finite dimension over $R = Z_{28}$.

$P_2 = \{\{0, 4, 8, 12, 16, 20, 24\}, +\} \subseteq V$ is also a S-vector subspace of V over $R = Z_{28}$.

$P_3 = \{\{0, 7, 14, 21\}, +\} \subseteq V$ is a S-vector subspace of V over $R = Z_{28}$.

$P_4 = \{\{0, 14\}, +\} \subseteq V$ is a S-vector subspace of V over $R = Z_{28}$.

All the four subspaces are of finite dimension over Z_{28}.

$T_1 = \{\{0, 0.5, 1, 1.5, 2, 2.5, 3, \ldots, 27, 27.5\}, +\} \subseteq V$ is a S-vector subspace of V over $R = Z_{28}$.

$T_2 = \{\{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6, 1.8, 2, \ldots, 27, 27.2, 27.4, 27.6, 27.8\}, +\} \subseteq V$ is a S-vector subspace of V over the S-ring $R = Z_{28}$.

Example 3.11: Let V = {[0, 26), +} be the S-special interval vector space over the S-ring R = Z_{26}.

This has S-subspaces of both finite and infinite dimension.

If on the S-special interval vector space V over the S-ring we can define a compatible product × on V then we define V to be a S-special interval pseudo linear algebra over the S-ring Z_n, since $a \times (b + c) \neq a \times b + a \times c$ for all a, b, c ∈ V.

We see V has substructure which are not linear subalgebras only quasi vector subspaces of V over the S-ring Z_n.

We will illustrate this situation by some examples.

Example 3.12: Let V = {[0, 22), +, ×} be a S-special pseudo interval linear algebra over the S-ring Z_{22} = R. Clearly V has S-quasi vector subspaces given by

$P_1 = \{\{0, 1, 2, \ldots, 21\}, +, \times\} \subseteq V$ is a S-special interval linear pseudo subalgebra of V over R.

$P_2 = \{\{0, 0.5, 1.0, 1.5, 2, 2.5, \ldots, 20, 20.5, 21, 21.5\}, +, \times\} \subseteq V$ is a S-special quasi vector subspace of V over R = Z_{22}.

$P_3 = \{\{0, 0.1, 0.2, \ldots, 0.9, 1, 1.1, 1.2, \ldots, 1.9, 2, \ldots, 20, 20.1, \ldots, 21, 21.1, 21.2, \ldots, 21.9\}, +, \times\} \subseteq V$ is a S-special quasi vector subspace of V over R = Z_{22}. Thus we have several S-special quasi vector subspaces of V over R = Z_{22}.

Now $P_4 = \{\{0, 2, 4, 6, \ldots, 20\}, +, \times\} \subseteq V$ is a S-special linear pseudo subalgebra of V over R = Z_{22}.

Similarly $P_5 = \{\{0, 11\}, +, \times\} \subseteq V$ is a S-special pseudo linear subalgebra of V over R = Z_{22}.

Example 3.13: Let V = {[0, 24), +, ×} be a S-special pseudo linear algebra over the S-ring R = Z_{24}. V has finite dimensional

S-quasi vector subspaces of finite dimension as well as finite dimensional S-special pseudo linear subalgebras.

Let $M_1 = \{\{0, 0.5, 0.15, 2, \ldots, 23, 23.5\}, +, \times\} \subseteq V$ be a S-quasi vector subspace of V over R of finite dimension over R.

$M_2 = \{\{0, 0.2, 0.4, \ldots, 1, 1.2, 1.4, \ldots, 2, 2.2, 2.4, \ldots, 3, 23, 23.2, \ldots, 23.8\}, +, \times\} \subseteq V$ is a S-quasi subvector space of V over the S-ring Z_{24}.

$M_3 = \{\{0, 1, 2, \ldots, 23\}, +, \times\} \subseteq V$ is a S-special pseudo linear subalgebra of V over the S-ring Z_{24}.

$M_4 = \{\{0, 2, 4, 6, 8, \ldots, 22\}, +, \times\} \subseteq V$ is a S-special pseudo linear subalgebra of V over the S-ring Z_{24}.

$M_5 = \{\{0, 3, 6, 9, 12, 15, 18, 21\}, +, \times\} \subseteq V$ is a S-special linear pseudo subalgebra of V over the S-ring Z_{24}.

$M_6 = \{\{0, 4, 8, 12, 16, 20\}, +, \times\} \subseteq V$ is a S-special pseudo linear subalgebra of V over the S-ring Z_{24}.

All these S-special quasi vector subspaces as well as all the S-special linear pseudo subalgebra of V over the ring R.

Example 3.14: Let $V = \{[0, 105), +, \times\}$ be the S-special pseudo linear algebra over the S-ring $R = Z_{105}$. V has S-special quasi vector subspaces of finite dimension one $R = Z_{105}$ which is as follows:

$N_1 = \{\{0, 0.5, 1, 1.5, \ldots, 104, 104.5\}, +, \times\} \subseteq V$ is only a S-special quasi vector subspace of V over R. Clearly N_1 is of finite dimension over R.

$N_2 = \{\{0, 1, 2, \ldots, 105\}, +, \times\} \subseteq V$ is a S-special pseudo linear subalgebra of V over $R = Z_{105}$.

Clearly dimension of N_2 over R is one.

$N_3 = \{\{0, 5, 10, 15, 20, \ldots, 100\}, +, \times\} \subseteq V$ is a S-special linear subalgebra of V over $R = Z_{105}$. Clearly N_3 over R is finite dimensional.

$N_4 = \{\{0, 3, 6, 9, 12, 15, \ldots, 102\}, +, \times\} \subseteq V$ is a S-special linear subalgebra of V over R.

$N_5 = \{\{0, 0.1, 0.2, \ldots, 0.9, 1, 1.1, 1.2, \ldots, 1.9, 2, \ldots, 104, 104.1, 104.2, \ldots, 104.9\}, +, \times\} \subseteq V$ is a S-special quasi vector subspace of V over $R = Z_{105}$.

Example 3.15: Let
$V = \{(a_1, a_2, a_3, a_4) \mid a_i \in [0, 22), 1 \leq i \leq 4, +\}$ be a S-special interval vector space over the S-ring $R = Z_{22}$. V has several S-special interval vector subspaces some of which are finite dimensional are some of them are infinite dimensional over S-ring $R = Z_{22}$.

$M_1 = \{(a_1, 0, 0, 0) \mid a_1 \in [0, 22), +\} \subseteq V$ is a S-special interval vector subspace of V over the S-ring $R = Z_{22}$.

$M_2 = \{(0, a_2, 0, 0) \mid a_2 \in [0, 22), +\} \subseteq V$ be the S-special interval vector subspace of V over the S-ring $R = Z_{22}$.

$M_3 = \{(0, 0, a_3, 0) \mid a_3 \in [0, 22), +\} \subseteq V$ be the S-special interval vector subspace of V over the S-ring $R = Z_{22}$.

$M_4 = \{(0, 0, 0, a_4) \mid a_4 \in [0, 22), +\} \subseteq V$ be the S-special interval vector subspace of V over the S-ring $R = Z_{22}$.

Clearly $V = M_1 + M_2 + M_3 + M_4$ is a direct sum and $M_i \cap M_j = \{(0, 0, 0, 0)\}$; $i \neq j$; $1 \leq i, j \leq 4$.

All of four spaces are infinite dimensional over $R = Z_{22}$.

Let $T_1 = \{(a_1, 0, 0, 0) \mid a_1 \in \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 20, 20.5, 21, 21.5\}, +\} \subseteq V$ be a S-special subspace of V over the S-ring $R = Z_{22}$ of finite dimension over $R = Z_{22}$.

Let $P_1 = \{(a_1, 0, 0, a_2) \mid a_2 \in [0, 22), a_1 \in \{0, 0.1, 0.2, \ldots, 0.9, 1, 1.1, \ldots, 20, 20.1, \ldots, 20.9, 21, 21.1, \ldots, 21.9\} \subseteq \{[0, 22), +\} \subseteq V$ be a S-special interval subspace of V over the S-ring $R = Z_{22}$.

We see $P_2 = \{(0, a_1, a_2, 0) \mid a_1, a_2 \in [0, 22), +\} \subseteq V$ is a special vector subspace of V over the S-ring Z_{22}.

$P_3 = \{(0, 0, a_1, a_2) \mid a_1 \in Z_2, a_2 = \{(0, 0.1, 0.2, \ldots, 1, 1.1, 1.2, \ldots, 20, 20.1, \ldots, 20.9, 21, \ldots, 21.9\} \subseteq [0, 22), +\} \subseteq V$ is a S-special vector subspace of V over the S-ring Z_{22}.

All these subspaces P_3, P_1 and T_1 cannot be made into a S-special pseudo linear subalgebras only a S-special vector subspace of V over the S-ring Z_{22}.

Example 3.16: Let

$$V = \left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}\right| a_i \in [0, 24), 1 \leq i \leq 6, +, \times_n\right\}$$

is a S-special pseudo linear algebra over the S-ring Z_{24}.

V has several S-special pseudo sublinear algebras.

$$T_1 = \left\{\left.\begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right| a_1 \in [0, 24), +, \times_n\right\} \subseteq V.$$

$$T_2 = \left\{ \left. \begin{bmatrix} 0 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right| a_2 \in [0, 24), +, \times_n \right\} \subseteq V,$$

$$T_3 = \left\{ \left. \begin{bmatrix} 0 \\ 0 \\ a_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right| a_3 \in [0, 24), +, \times_n \right\} \subseteq V,$$

$$T_4 = \left\{ \left. \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \\ 0 \\ 0 \end{bmatrix} \right| a_2 \in [0, 24), +, \times_n \right\} \subseteq V,$$

$$T_5 = \left\{ \left. \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_5 \\ 0 \end{bmatrix} \right| a_5 \in [0, 24), +, \times_n \right\} \subseteq V \text{ and}$$

$$T_6 = \left\{ \left. \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_6 \end{bmatrix} \right| a_6 \in [0, 24), +, \times_n \right\} \subseteq V$$

are S-special interval pseudo sublinear algebras of V over F.

Clearly $T_i \cap T_j = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ if $i \neq j$, $1 \leq i, j \leq 6$.

Further $V = T_1 + T_2 + T_3 + T_4 + T_5 + T_6$ is a direct sum of sublinear pseudo algebras of V over $R = Z_{24}$.

Let

$$P_1 = \left\{ \left. \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right| a_1 \in \{0, 0.1, 0.2, \ldots, 0.9, 1, 1.1, 1.2, \ldots, 23, 23.1, \ldots, 23.9\} \subseteq [0, 24); + \right\} \subseteq V$$

is only a S-special quasi vector subspace of V and not a S-pseudo linear subalgebra of V over the S-ring R.

We have several such S-special quasi vector subspaces which are finite dimensional over R.

We see even in case of S-special interval pseudo linear algebras we have S-subalgebras or S-quasi subvector spaces of V such that they are orthogonal with each other.

Let

$$B_1 = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right| a_1, a_2 \in [0, 24), +, \times_n \right\} \subseteq V$$

be the S-special pseudo linear subalgebra of V.

$$B_2 = \left\{ \left. \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \end{bmatrix} \right| a_1, a_2 \in \{0, 0.5, 1, 1.5, \ldots, 23, 23.5, \subseteq V, +\} \right. \subseteq V$$

be the S-special quasi vector subspace of V over the S-ring $R = Z_{24}$.

$$\text{We see } B_1 \times B_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

That is B_1 is orthogonal to B_2 and vice versa. However B_2 is finite dimensional where as B_1 is infinite dimensional B_1 is a S-sublinear pseudo algebra but B_2 is a S-quasi vector subspace of V.

Also $B_1 + B_2 \neq V$ only a special quasi vector subspace of V over R.

$$\text{Infact take } B_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| \; a_i \in \{0, 0.5, 1, 1.5, \ldots, 23, 23.5\} \subseteq \right.$$

$[0, 24)$; $1 \leq i \leq 4, +\} \subseteq V$ is only a S-special quasi vector subspace of V over the S-ring $R = Z_{24}$.

$$\text{We see if } x = \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ 0 \\ 0.5 \\ 1.5 \end{bmatrix} \text{ and } y = \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ 1.5 \\ 1.5 \\ 0.5 \end{bmatrix} \text{ are in } B_3;$$

$$\text{then } x \times_n y = \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ 0 \\ 0.5 \\ 1.5 \end{bmatrix} \times_n \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ 1.5 \\ 1.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.25 \\ 0 \\ 0.75 \\ 0.75 \end{bmatrix} \notin B_3.$$

Thus B_3 is only a S-special quasi vector subspace of V over R. Further B_3 is orthogonal with B_1 but B_3 is not orthogonal with B_2.

B_1 is also orthogonal with B_3; B_3 only a S-special quasi vector subspace of V over R.

Example 3.17: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \ a_i \in [0, 28); 1 \leq i \leq 9, +, \times \right\}$$

be a S-special interval pseudo linear algebra over the S-ring R = Z_{28}. We have S-quasi special vector subspaces of finite dimension over Z_{28}. However V is a non commutative S-special pseudo interval linear algebra.

Take

$$M_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \ a_i \in \{0, 0.5, 1, 1.5, 2, \ldots, 27, 27.5\} \subseteq Z_{28} \right\} \subseteq V$$

is only a S-special quasi vector subspace of V over the S-ring Z_{28}. The dimension of M_1 over V is finite dimensional over R = Z_{28}.

Let

$$M_2 = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ a_4 & a_5 & a_6 \end{bmatrix} \right| a_i \in [0, 28), 1 \leq i \leq 3 \text{ and } a_j \in \right.$$

$$\{0, 0.1, 0.2, 0.3, \ldots, 1, 1.1, \ldots, 2, 27.1, 27.2, \ldots, 27.9\} \subseteq V,$$

$$\left. 4 \leq j \leq 6; +, \times \right\} \subseteq V;$$

M_2 is a S-special interval quasi vector subspace of V over $R = Z_{28}$. Clearly M_2 is a infinite dimensional subspace of V over $Z_{28} = R$.

We see for B, A $\in M_2$ in general $A \times B \notin M_2$.

We show V is a S-interval non commutative special interval pseudo linear algebra.

$$\text{Let } A = \begin{bmatrix} 0.5 & 3 & 4 \\ 1 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 0.5 \\ 0 & 1 & 2 \end{bmatrix} \in M;$$

$$A \times B = \begin{bmatrix} 0.5 & 3 & 4 \\ 1 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 0.5 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0.5 & 17 & 9.5 \\ 1 & 10 & 1 \\ 0 & 8 & 8.5 \end{bmatrix} \ldots \text{ I}$$

Consider

$$B \times A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 0.5 \\ 0 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 0.5 & 3 & 4 \\ 1 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2.5 & 7 & 4 \\ 4 & 8.5 & 2 \\ 1 & 4 & 8 \end{bmatrix} \ldots \text{II.}$$

Clearly I and II are distinct. So V is a S-special interval non commutative pseudo linear algebra under usual matrix multiplication.

Let $T_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \middle| \; a_i \in \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 27,\right.$

$27.5\} \subseteq V;\ 1 \leq i \leq 3; +, \times\} \subseteq V$ be a S-special interval quasi vector subspace of V and is not a pseudo linear algebra.

Further T_1 is finite dimensional subspace of V over $F = Z_{28}$.

Let

$$A = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 2.5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \in T_1.$$

$$A \times B = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 2.5 \end{bmatrix} \times \begin{bmatrix} 6.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$= \begin{bmatrix} 3.25 & 0 & 0 \\ 0 & 0.75 & 0 \\ 0 & 0 & 3.75 \end{bmatrix} \notin T_1.$$

Thus T_1 is only a finite dimensional S-subspace.

Example 3.18: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i \in [0, 6); 1 \leq i \leq 9, +, \times_n \right\}$$

be the S-special interval pseudo linear algebra over the ring $R = Z_6$.

V is a commutative S-special interval pseudo linear algebra of infinite order.

Let

$$M_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 0 \end{bmatrix} \middle| a_i \in [0, 6); 1 \leq i \leq 6, +, \times_n \right\} \subseteq V;$$

M_1 is a S-special interval pseudo linear subalgebra of V over R $= Z_6$.

$$M_2 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \middle| a_i \in \{0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5, 5.5\} \subseteq Z_6; 1 \leq i \leq 3, +, \times_n \right\} \subseteq V$$

be the S-special interval quasi vector subspace of V over $F = Z_6$.

Clearly M_2 is not a S-special pseudo linear algebra is only a S-special quasi vector subspace over V.

Let $A, B \in M_2$ where

$$A = \begin{bmatrix} 0.5 & 0 & 0 \\ 2.5 & 0 & 0 \\ 0 & 0 & 3.5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \in M_2.$$

We find

$$A \times_n B = \begin{bmatrix} 0.5 & 0 & 0 \\ 2.5 & 0 & 0 \\ 0 & 0 & 3.5 \end{bmatrix} \times_n \begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.25 & 0 & 0 \\ 1.25 & 0 & 0 \\ 0 & 0 & 1.75 \end{bmatrix} \notin M_2.$$

Clearly M_2 is not a S-linear pseudo subalgebra only a S-quasi vector subspace of V over R = Z_6. M_2 is commutative and finite dimensional over R = Z_6.

Let $M_3 = \left\{ \left. \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ a_3 & 0 & 0 \end{bmatrix} \right| a_i \in \{0, 0.1, 0.2, \ldots, 0.9, 1, 1.1,\right.$

1.2, …, 5, 5.1, 5.2, …, 5.9}$\subseteq Z_6$, $1 \leq i \leq 3$, +, $\times_n$} $\subseteq$ V is again a S-special quasi vector subspace of V over the S-ring Z_6.

Let

$$A = \begin{bmatrix} 0.2 & 0 & 0 \\ 0.1 & 0 & 0 \\ 0.9 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2.2 & 0 & 0 \\ 5.8 & 0 & 0 \\ 6.4 & 0 & 0 \end{bmatrix} \in M_3;$$

$$\text{we see } A \times_n B = \begin{bmatrix} 0.2 & 0 & 0 \\ 0.1 & 0 & 0 \\ 0.9 & 0 & 0 \end{bmatrix} \times_n \begin{bmatrix} 2.2 & 0 & 0 \\ 5.8 & 0 & 0 \\ 6.4 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.44 & 0 & 0 \\ 0.58 & 0 & 0 \\ 5.76 & 0 & 0 \end{bmatrix} \notin M_3.$$

So M_3 is only a S-special interval quasi vector subspace of V over $R = Z_6$.

Clearly M_3 is finite dimensional over $R = Z_6$ however M_3 is not a special interval pseudo linear subalgebra over Z_6.

Let

$$M_4 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_1 & 0 \\ a_2 & 0 & a_3 \end{bmatrix} \middle| \; a_i \in [0, 6), 1 \leq i \leq 3 \right\} \subseteq V$$

be the S-special interval pseudo linear algebra over the S-ring Z_6. M_4 is also infinite dimensional over the S-ring $R = Z_6$.

Consider

$$P_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1 \in [0, 6), +, \times_n \right\} \subseteq V,$$

$$P_2 = \left\{ \begin{bmatrix} 0 & a_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_2 \in [0, 6), +, \times_n \right\} \subseteq V,$$

$$P_3 = \left\{ \begin{bmatrix} 0 & 0 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_3 \in [0, 6), +, \times_n \right\} \subseteq V,$$

$$P_4 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ a_4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right| a_4 \in [0, 6), +, \times_n\right\} \subseteq V,$$

$$P_5 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ 0 & a_5 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right| a_5 \in [0, 6), +, \times_n\right\} \subseteq V, \ldots, \text{ and}$$

$$P_9 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_9 \end{bmatrix}\right| a_9 \in [0, 6), +, \times_n\right\} \subseteq V$$

are the nine S-special interval pseudo linear subalgebras of V over
$R = Z_6$.

We see $P_i \cap P_j = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}$ if $i \neq j$, $1 \leq i, j \leq 9$.

However $V = B_1 + B_2 + B_3 + B_4 + B_5 + B_6 \subseteq V$.

Hence is not a direct sum of pseudo sublinear algebras over Z_6.

Example 3.19: Let

$$V = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix}\right| a_i \in [0, 12), 1 \leq i \leq 30, +, \times_n\right\}$$

be the S-special pseudo linear algebra over the S-ring Z_{12}.

V has both quasi vector subspaces and pseudo sublinear algebras of finite dimension as well as infinite dimension over the S-ring $R = Z_{12}$.

Let

$$A_1 = \left\{\left.\begin{bmatrix} a_1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}\right| a_1 \in [0, 12), +, \times_n\right\} \subseteq V,$$

$$A_2 = \left\{\left.\begin{bmatrix} 0 & a_2 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}\right| a_2 \in [0, 12), +, \times_n\right\} \subseteq V,$$

$$A_3 = \left\{\left.\begin{bmatrix} 0 & 0 & a_3 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}\right| a_3 \in [0, 12), +, \times_n\right\} \subseteq V \text{ and so on.}$$

$$A_{27} = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & a_{27} \\ 0 & 0 & 0 \end{bmatrix}\right| a_{27} \in [0, 12), +, \times_n\right\} \subseteq V,$$

$$A_{28} = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ a_{28} & 0 & 0 \end{bmatrix}\right| a_{28} \in [0, 12), +, \times_n\right\} \subseteq V,$$

$$A_{29} = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & a_{29} & 0 \end{bmatrix}\right| a_{29} \in [0, 12), +, \times_n\right\} \subseteq V \text{ and}$$

$$A_{30} = \left\{\left[\begin{array}{ccc} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & a_{30} \end{array}\right] \middle| \ a_{30} \in [0, 12), +, \times_n\right\} \subseteq V \text{ are}$$

S-special interval pseudo linear subalgebras of V over the S-ring Z_{12}.

Clearly

$$A_i \cap A_j = \left\{\left[\begin{array}{ccc} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]\right\} \text{ if } i \neq j,\ 1 \leq i, j \leq 30$$

and $A_1 + A_2 + \ldots + A_{30} = V$ is the direct sum of S-special interval pseudo linear subalgebras of V. Every A_i is infinite dimensional over $R = Z_{12}$.

We have 30 sublinear algebra of finite dimension but they will not lead to the direct sum.

$$\text{Let } B_1 = \left\{\left[\begin{array}{ccc} a_1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{array}\right] \middle| \ a_1 \in Z_{12}, +, \times_n\right\} \subseteq V,$$

$$B_2 = \left\{\left[\begin{array}{ccc} 0 & a_2 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{array}\right] \middle| \ a_2 \in Z_{12}, +, \times_n\right\} \subseteq V,$$

$$B_3 = \left\{\left[\begin{array}{ccc} 0 & 0 & a_3 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{array}\right] \middle| \ a_3 \in Z_{12}, +, \times_n\right\} \subseteq V, \ldots,$$

$$B_{28} = \left\{ \left. \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ a_{28} & 0 & 0 \end{bmatrix} \right| a_{28} \in Z_{12}, +, \times_n \right\} \subseteq V,$$

$$B_{29} = \left\{ \left. \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & a_{29} & 0 \end{bmatrix} \right| a_{29} \in Z_{12}, +, \times_n \right\} \subseteq V \text{ and}$$

$$B_{30} = \left\{ \left. \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & a_{30} \end{bmatrix} \right| a_{30} \in Z_{12}, +, \times_n \right\} \subseteq V$$

be S-special interval pseudo linear subalgebras of V.

All of them are one dimensional over Z_{12}.
But $B_1 + \ldots + B_{30} \neq V$ however

$$B_i \cap B_j = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \text{ if } i \neq j,\ 1 \leq i, j \leq 30.$$

Likewise we can have infinite dimensional S-special and finite dimensional quasi vector spaces.

$$D_1 = \left\{ \left. \begin{bmatrix} a_1 & a_2 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \right| a_1 \in [0, 12),\ a_2 \in \{0, 0.5, 1, 1.5, 2, \ldots, 11, 11.5\}, +, \times_n \right\} \subseteq V,$$

$$D_2 = \left\{ \left. \begin{bmatrix} 0 & 0 & a_3 \\ a_4 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \right| a_3 \in [0, 12), a_4 \in \{0, 0.2, 0.4, 0.6, 0.8, 1,\right.$$

$$\left. 1.2, \ldots, 11.2, 11.4, \ldots, 11.8\} \subseteq [0, 12), +, \times_n \right\} \subseteq V \text{ and so on.}$$

$$D_{14} = \left\{ \left. \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & a_{27} \\ a_{28} & 0 & 0 \end{bmatrix} \right| a_{27} \in [0, 12), a_4 \in \{0, 0.1, 0.2, \ldots, 1, 1.2,\right.$$

$$\left. \ldots, 11, 11.1, \ldots, 11.9\} \subseteq [0, 12), +, \times_n \right\} \subseteq V \text{ and}$$

$$D_{15} = \left\{ \left. \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & a_{29} & a_{30} \end{bmatrix} \right| a_{29} \in [0, 12) \text{ and } a_{30} \in \{0, 0.5, 1, 1.5, \ldots,\right.$$

$$\left. 11, 11.5\} \subseteq [0, 12), +, \times_n \right\} \subseteq V$$

be the S-special interval quasi vector subspaces of V over Z_{12}.

None of them is a special interval pseudo linear subalgebra as product is not defined in D_i for the (D_i, $\times_n$) is not a semigroup that there exists A, B $\in D_i$ such that A $\times_n$ B $\notin D_i$, thus all D_i's are only S-quasi vector subspaces and they are not pseudo linear subalgebras.

However

$$D_i \cap D_j = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \text{ for } i \neq j,\ 1 \leq i, j \leq 15.$$

Further $D_1 + \ldots + D_{15} \neq V$ so it is not a direct sum. All the 15 quasi vector subspaces are infinite dimensional over Z_{12}.

Let

$$W_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \right| a_1 \in \{0, 0.5, 1, 1.5, \ldots, 11, 11.5\} \subseteq$$

$$[0, 12), +, \times_n\} \subseteq V,$$

$$W_2 = \left\{ \begin{bmatrix} 0 & a_2 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \right| a_2 \in \{0, 0.5, 1, 1.5, \ldots, 11, 11.5\} \subseteq$$

$$[0, 12), +, \times_n\} \subseteq V,$$

$$W_3 = \left\{ \begin{bmatrix} 0 & 0 & a_3 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \right| a_3 \in \{0, 0.5, 1, 1.5, \ldots, 11, 11.5\} \subseteq [0, 12),$$

$$+, \times_n\} \subseteq V \text{ and so on.}$$

$$W_{30} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & a_{30} \end{bmatrix} \right| a_{30} \in \{0, 0.5, 1, 1.5, \ldots, 11, 11.5\} \subseteq [0,$$

$$12), +, \times_n\} \subseteq V$$

be S-special interval linear quasi vector subspace of V over $R = Z_{12}$. Clearly each W_i is finite dimensional over $R = Z_2$;

$$W_i \cap W_j = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \text{ if } i \neq j,\ 1 \leq i, j \leq 30.$$

Further $W = W_1 + \ldots + W_{30} \subseteq V$ so is not a direct sum we see

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \middle| a_2 \in \{0, 0.5, 1, 1.5, \ldots, 11, 11.5\} \subseteq \right.$$

$$\left. [0, 12), +, \times_n, 1 \leq i \leq 30 \right\} \subseteq V,$$

W is a S-quasi subvector space of V and W is the subdirect subsum of W_i's, $1 \leq i \leq 30$.

Example 3.20: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & \ldots & \ldots & a_8 \\ a_9 & \ldots & \ldots & a_{12} \\ a_{13} & \ldots & \ldots & a_{16} \\ a_{17} & \ldots & \ldots & a_{20} \\ a_{21} & \ldots & \ldots & a_{24} \end{bmatrix} \middle| a_i \in [0, 15), 1 \leq i \leq 24, +, \times_n \right\}$$

be a S-special interval pseudo linear algebra over the S-ring Z_{15}. V has finite dimensional S-sublinear pseudo algebras and infinite dimensional S-linear pseudo subalgebras of V.

Infact V has finite dimensional S-quasi vector subspaces as well as infinite dimensional S-quasi vector subspaces.

Further V can be represented as a direct sum of sublinear algebras.

Let

$$M_1 = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \right| a_i \in [0, 15), 1 \leq i \leq 4, +, \times_n \right\} \subseteq V$$

be a special interval linear pseudo subalgebra of infinite dimension over $R = Z_{15}$.

$$M_2 = \left\{ \left. \begin{bmatrix} 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \right| a_i \in [0, 15), 1 \leq i \leq 4, +, \times_n \right\} \subseteq V$$

is again a S-special interval pseudo linear subalgebra of V of infinite dimension over $R = Z_{15}$.

Let

$$M_3 = \left\{ \left. \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \right| a_i \in [0, 15), 1 \leq i \leq 4, +, \times_n \right\} \subseteq V$$

be a S-special interval pseudo linear subalgebra of V the S-ring $R = Z_{15}$.

$$M_4 = \left\{ \left. \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \right| a_i \in [0, 15), 1 \leq i \leq 4, +, \times_n \right\} \subseteq V$$

is again a S-special interval pseudo linear subalgebra of V over $R = Z_{15}$.

$$M_5 = \left\{ \left. \begin{bmatrix} 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right| a_i \in [0, 15), 1 \leq i \leq 4, +, \times_n \right\} \subseteq V.$$

$$M_6 = \left\{ \left. \begin{bmatrix} 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \end{bmatrix} \right| a_i \in [0, 15), 1 \leq i \leq 4, +, \times_n \right\} \subseteq V$$

is again a S-special interval pseudo linear subalgebra of infinite dimension over $R = Z_{15}$.

All the six S-sublinear pseudo algebras are infinite dimension over $R = Z_{15}$.

$$\text{Further } M_i \cap M_j = \left\{\begin{bmatrix} 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right\} \text{ if } i \neq j, 1 \leq i, j \leq 6.$$

Also V = $M_1 + M_2 + \ldots + M_6$ is the direct sum of S-sublinear pseudo algebras.

However V can be represented as a direct sum in several ways.

We will now proceed onto describe S-special interval pseudo linear subalgebras of finite dimension over R = Z_{15}.

$$\text{Let } T_1 = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| a_i \in Z_{15}, 1 \leq i \leq 4, +, \times_n\right\} \subseteq V,$$

$$T_2 = \left\{\begin{bmatrix} 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| a_i \in Z_{15}, 1 \leq i \leq 4, +, \times_n\right\} \subseteq V,$$

$$T_5 = \left\{\begin{bmatrix} 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| a_i \in Z_{15}, 1 \leq i \leq 4, +, \times_n\right\} \subseteq V$$

and

$$T_6 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \end{bmatrix} \middle| a_i \in Z_{15}, 1 \leq i \leq 4, +, \times_n \right\} \subseteq V$$

are the six S-special pseudo linear subalgebras of V over R = Z_{15}.

We see all the S-sublinear pseudo algebras are of dimension four over R = Z_{15}. But V ≠ $T_1 + \ldots + T_6$ that is V cannot be written as a direct sum of T_i's $1 \leq i \leq 6$.

Similarly V can have both finite and infinite dimensional S-special interval quasi vector subspaces of V which are not linear algebras of V over the S-ring Z_{15}.

Let

$$B_1 = \left\{ \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| a_1 \in [0, 15), a_2 \in \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 14, 14.5\} \subseteq [0, 15), +, \times_n \right\} \subseteq V$$

be a S-vector subspace and not a S-linear subalgebra of V.

For let

$$x = \begin{bmatrix} 9 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 3 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \in B_1.$$

We find

$$x \times_n y = \begin{bmatrix} 9 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \times_n \begin{bmatrix} 3 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \notin B_1,$$

hence B_1 is not a S-special interval pseudo linear subalgebra of V over R Z_{15}. The dimension of V over R is infinite dimensional as a S-quasi vector subspace over R = Z_{15}.

Let

$$D_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| \, a_i \in \{0, 0.1, 0.2, \ldots, 0.9, 1, 1.1, 1.2,\right.$$

$$\ldots, 1.9, 2, 2.1, \ldots, 2.9, \ldots, 14, 14.1, 14.2, \ldots, 14.9\} \subseteq [0, 15), +, \times_n \subseteq V$$

be a S-special interval quasi vector subspace over R = Z_{15} it is not a S-linear subalgebra over the S-ring R = Z_{15}.

For if

$$x = \begin{bmatrix} 0.1 & 0.4 & 1.2 & 0.8 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 1.6 & 2.2 & 1.1 & 4.2 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \in D_1$$

we find

$$x \times_n y = \begin{bmatrix} 0.1 & 0.4 & 1.2 & 0.8 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \times_n \begin{bmatrix} 1.6 & 2.2 & 1.1 & 4.2 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.16 & 0.88 & 1.32 & 3.36 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \notin D_1$$

as the entries do not belong to $\{0, 0.1, 0.2, 0.3, \ldots, 1, 1.1, \ldots, 14, 14.1, 14.2, \ldots, 14.9\} \subseteq [0, 15)$.

Hence D_1 is not closed under product so is only a S-special quasi vector subspace of V over $R = Z_{15}$. However D_1 is a finite dimensional S-special interval vector subspace of V over the S-ring R.

We have studied the notion of S-sublinear algebras of finite and infinite dimension and S-special pseudo linear subalgebras of finite dimension of V over the S-ring.

We can always write V as a direct sum of S-sublinear algebras over Z_{15}.

Further V has S-special interval quasi vector subspaces of both finite and infinite dimension over the S-ring Z_n.

Now we proceed onto study the S-special linear transformation and S-special linear operator of a S-special interval linear algebra (vector space) over a S-ring Z_n. We will also define S-special linear functional and so on. We can as in case of usual vector spaces define linear transformation in case of S-special interval vector spaces only if they are defined on the same S-ring Z_n.

Here we are going to define also the concept of special quasi induced linear transformation.

All these will be illustrated by some examples.

Example 3.21: Let $V = \{(a_1, a_2, a_3) \mid a_i \in [0, 21), 1 \leq i \leq 3, +\}$ be a S-special interval vector space over the S-ring Z_{21}.

$$W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \end{bmatrix} \middle| \; a_i \in [0, 21), 1 \leq i \leq 6, +\right\}$$

be a S-special interval vector space over the S-ring $R = Z_{21}$.

Define $T_1 : V \rightarrow W$ by

$$T_1\{(a_1, a_2, a_3)\} = \begin{bmatrix} a_1 \\ 0 \\ a_2 \\ 0 \\ a_3 \\ 0 \end{bmatrix} \text{ for every } (a_1, a_2, a_3) \in V.$$

Clearly T_1 is a S-special linear transformation of V to W.

We see ker $T_1 = \{(0, 0, 0)\}$. In this way we can define several such S-special linear transformation from V to W.

We can have another S-special linear transformation T_2 from V to W as follows.

$T_2 : V \to W$ by

$$T_2 \{(a_1, a_2, a_3)\} = \begin{bmatrix} a_1 + a_2 \\ a_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Clearly T_2 is also a S-special linear transformation from V to W and ker $T_2 \neq \{(0, 0, 0)\}$.

$$\text{For ker } T_2 = \left\{x = (a_1, a_2, a_3) \in V \mid T_2(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}.$$

$$\text{We see } T \{(a_1, a_2, a_3)\} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ if } a_1 + a_2 \equiv 0 \pmod{21} \text{ and } a_3 = 0.$$

Thus ker T ≠ {(0, 0, 0) and
ker T = $\{(a_1, a_2, a_3) \mid a_1 + a_2 = 0 \pmod{21}$ and $a_3 = 0\}$.

Interested reader can define may transformation of this form.

We can define the S-special linear transformation from W to V as follows.

$$S_1\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}\} = (a_1 + a_2,\ a_3 + a_4, a_5 + a_6).$$

S_1 is a S-special linear transformation from W to V.

Further

$$\ker S_1 = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}\right| a_1 + a_2 = 0, a_3 + a_4 = 0, a_5 + a_6 = 0, a_i \in [0, 21),$$

$$1 \le i \le 6\} \ne \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ as } a_1 = 20 \text{ and } a_2 = 1, a_1 = 5 \text{ and } a_2 = 16$$

and so on like $a_1 = 0.0003$ and $a_2 = 20.9997$, $a_3 + a_4 = 0$ where $a_3 = 19.2$ and $a_4 = 1.8$ $a_3 = 10$ and $a_4 = 11$ and so on, $a_5 = 0.04$ and $a_6 = 20.46$ and so on.

Thus ker S_1 is non trivial; ker S_1 is a S-subspace of V.

For we have to show

1. if $x, y \in \ker S_1$, $x + y \in \ker S_1$
2. if $x \in \ker S_1$ then $-x \in \ker S_1$
3. if $c \in Z_{21}$ and $x \in \ker S_1$, $cx \in \ker S_1$.

All the three conditions can be easily proved without any difficulty. Hence the claim.

Suppose we define $S_2 : W \to V$ by

$$S_2\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}\} = (a_1, a_2, a_3)$$

be a S-special linear transformation of W to V.

$$\text{Then also } \ker S_2 \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\ker S_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| a_i \in [0, 21); 4 \leq i \leq 6 \right\} \subseteq W$$

is a proper subspace of W over $R = Z_{21}$.

Now S_1 and S_2 are S-special linear transformations from W to V.

Example 3.22: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} \middle| a_i \in [0, 93), 1 \leq i \leq 16 \right\}$$

be the S-special subset interval vector space over the S-ring $R = Z_{93}$.

V can be written as a direct sum of S-subvector spaces over $R = Z_{93}$.

V has both S-subspaces of finite and infinite dimension over $R = Z_{93}$; on V we can define linear operator.

The linear operator in this case also is the same as the usual spaces.

Let

$$T_1 : V \to V \text{ defined by } T_1\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix}\right\} = \begin{bmatrix} a_1 & 0 \\ 0 & a_4 \\ a_5 & 0 \\ 0 & a_6 \\ a_7 & 0 \\ 0 & a_8 \\ a_9 & 0 \\ 0 & a_{10} \end{bmatrix}.$$

It is easily verified T_1 is a linear operator on V and

$$\ker T_1 \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}.$$

Now define $T_2 : V \to V$

$$T_2\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix}\right\} = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \\ a_3 & a_4 \\ 0 & 0 \\ a_5 & a_6 \\ 0 & 0 \\ a_7 & a_8 \\ 0 & 0 \end{bmatrix}.$$

T_2 is also a linear operator on V. ker $T_2 \neq \{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}\}$

Consider T_1 o T_2 , T_1 o T_2 $\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix}\}$.

$$T_2 (T_1 \{\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix}\})$$

$$= T_2 (\begin{bmatrix} a_1 & 0 \\ 0 & a_4 \\ a_5 & 0 \\ 0 & a_6 \\ a_7 & 0 \\ 0 & a_8 \\ a_9 & 0 \\ 0 & a_{10} \end{bmatrix}) = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \\ a_5 & 0 \\ 0 & 0 \\ a_7 & 0 \\ 0 & 0 \\ a_9 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\text{Clearly } (T_1 \text{ o } T_2) \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix}\right\} = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \\ a_5 & 0 \\ 0 & 0 \\ a_7 & 0 \\ 0 & 0 \\ a_9 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\text{Consider } (T_2 \text{ o } T_1) \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix}\right\} =$$

$$T_1 \left\{T_2 \left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix}\right)\right\} = T_1 \left\{\begin{bmatrix} a_1 & a_2 \\ 0 & 0 \\ a_3 & a_4 \\ 0 & 0 \\ a_5 & a_6 \\ 0 & 0 \\ a_7 & a_8 \\ 0 & 0 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \\ a_3 & a_4 \\ 0 & 0 \\ a_5 & a_6 \\ 0 & 0 \\ a_7 & a_8 \\ 0 & 0 \end{bmatrix}.$$

We see in this case $T_1 \text{ o } T_2 = T_2 \text{ o } T_1$ and $T_1 \text{ o } T_2 : V \to V$ is again a linear operator on V.

Let $T_3 : V \to V$ defined by

$$T_3\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix}\} = \begin{bmatrix} a_1 & 0 \\ a_3 & 0 \\ \vdots & \vdots \\ a_{15} & 0 \end{bmatrix}; T_3 \text{ is a linear operator on V.}$$

$$T_3 \text{ o } T_1 (\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix}) = T_1 (T_3 \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix})$$

$$= T_1 \{\begin{bmatrix} a_1 & 0 \\ a_3 & 0 \\ \vdots & \vdots \\ a_{15} & 0 \end{bmatrix}\} = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \\ a_5 & 0 \\ 0 & 0 \\ a_7 & 0 \\ 0 & 0 \\ a_9 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\text{Consider } T_1 \text{ o } T_3 (\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix}) = T_3 (T_1 (\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix}))$$

$$= T_3 \left(\begin{bmatrix} a_1 & 0 \\ 0 & a_3 \\ a_5 & 0 \\ 0 & a_8 \\ a_9 & 0 \\ 0 & a_{10} \\ a_{11} & 0 \\ 0 & a_{12} \end{bmatrix} \right) = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \\ a_5 & 0 \\ 0 & 0 \\ a_9 & 0 \\ 0 & 0 \\ a_{11} & 0 \\ 0 & 0 \end{bmatrix}.$$

In this case also we see $T_1 \text{ o } T_3 = T_3 \text{ o } T_3$ and $T_1 \text{ o } T_3$ is a linear operator on V.

Now let

$$W_1 = \left\{ \begin{bmatrix} a_1 & 0 \\ a_2 & 0 \\ \vdots & \vdots \\ a_8 & 0 \end{bmatrix} \middle| \, a_i \in [0, 93), 1 \leq i \leq 8 \right\} \subseteq V$$

be a S-special interval vector subspace of V over the S-ring.

We define $T : V \to V$ by

$$T \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} \right\} = \begin{bmatrix} a_1 & 0 \\ a_3 & 0 \\ \vdots & \vdots \\ a_{15} & 0 \end{bmatrix};$$

clearly T is a linear operator which is a projection of V onto the subspace W_1.

$$\text{Now } T \text{ o } T(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix}) = T(\begin{bmatrix} a_1 & 0 \\ a_3 & 0 \\ \vdots & \vdots \\ a_{15} & 0 \end{bmatrix}).$$

$$= \begin{bmatrix} a_1 & 0 \\ a_3 & 0 \\ \vdots & \vdots \\ a_{15} & 0 \end{bmatrix}.$$

Thus T o T = T for (T o T) [(x)] = T(x) for all $x \in V$.

We can have the notion of projection of V onto a subspace of V.

Example 3.23: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \, a_i \in [0, 94), 1 \leq i \leq 12 \right\}$$

be a S-special interval vector space over the S-ring Z_{94}.

We see

$$P_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| \, a_1 \in [0, 94) \right\} \subseteq V,$$

$$P_2 = \left\{ \begin{bmatrix} 0 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| \, a_1 \in [0, 94) \right\} \subseteq V, \ldots,$$

$$P_{15} = \left\{\left.\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{15} & 0 \end{bmatrix}\right| a_{15} \in [0, 94) \right\} \subseteq V \text{ and}$$

$$P_{16} = \left\{\left.\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{16} \end{bmatrix}\right| a_{16} \in [0, 94) \right\} \subseteq V$$

be the 16 S-subspaces of V over the same S-ring Z_{16}.

We see if

$$T_1 : V \to V \text{ such that } T_1\{(\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix})$$

$$= \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is a special linear operator on V and it is a projection to the space P_1.

Likewise $T_{10} : V \to V$ given by

$$T_{10}\left(\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a_{10} & 0 & 0 \end{bmatrix};$$

T_{10} is a projection to the space P_{10}.

Thus we have $T_1, T_2, \ldots, T_{12}$ to be S-special linear operators which are all projections of V to the subspaces of P_i. Further $T : V \to V$ given by

$$T\left(\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix}\right) = \begin{bmatrix} a_1 & 0 & a_3 & 0 \\ 0 & a_6 & 0 & a_8 \\ a_9 & 0 & a_{11} & 0 \end{bmatrix}$$

is a linear operator on V; we see

$$ToT_2\left(\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix}\right) = T_2(T\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix})$$

$$= T_2\left(\begin{bmatrix} a_1 & 0 & a_3 & 0 \\ 0 & a_6 & 0 & a_8 \\ a_9 & 0 & a_{11} & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$T_2 \text{ o } T\{(\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix})\}$$

$$= T(T_2\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix})$$

$$= T\left(\begin{bmatrix} 0 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus $T \text{ o } T_2 = T_2 \text{ o } T = T_o$; the zero S-linear transformation of V to V.

We see $T_i \text{ o } T_j = T_o$ if $i \neq j$, and $T_i \text{ o } T_i = T_i$; $1 \leq i, j \leq 12$.

This is the way we get projections depending on the subspaces.

All projections may not satisfy $T_i \circ T_j = T_o$ $(i \neq j)$. This will not be true in the case of all S-subspaces of V.

For take

$$B_1 = \left\{\begin{bmatrix} a_1 & a_2 & 0 & a_3 \\ 0 & 0 & a_4 & 0 \\ 0 & 0 & a_5 & a_6 \end{bmatrix} \middle| a_i \in [0, 94), 1 \leq i \leq 6\right\} \subseteq V$$

is a S-subspace of B.

$$B_2 = \left\{\begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & a_4 \end{bmatrix} \middle| a_i \in [0, 94), 1 \leq i \leq 4\right\} \subseteq V$$

be the S-subspace of V over the S-ring $R = Z_{94}$.

$S_1 : V \to V$ and

$$S_1(\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix}) = \begin{bmatrix} a_1 & a_2 & 0 & a_3 \\ 0 & 0 & a_4 & 0 \\ 0 & 0 & a_5 & a_6 \end{bmatrix}$$

is a S-special linear operator which is also a projection of V to B_1.

Now consider

$$S_2 : V \to V \text{ be defined by } S_2(\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix})$$

$$= \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & a_4 \end{bmatrix};$$

S_2 is a S-special linear operator on V and is also a projection of V to B_2.

Now we see $S_1 \circ S_2 \neq S_o = T_o$; the zero S-linear operator on V.

$$\text{Consider } S_1 \circ S_2 \left(\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \right)$$

$$= S_2\{S_4 \left(\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \right)\}$$

$$= S_2\left(\begin{bmatrix} a_1 & a_2 & 0 & a_3 \\ 0 & 0 & a_4 & 0 \\ 0 & 0 & a_5 & a_6 \end{bmatrix} \right)$$

$$= \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & a_4 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\text{Consider } S_2 \circ S_1\left(\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \right)$$

$$= S_1\{S_2\left(\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \right)\}$$

$$= S_1 (\begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & a_4 \end{bmatrix}) = \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & a_4 \end{bmatrix}.$$

Let $W_1 : V \to V$ be defined by

$$W_1 (\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix}) = \begin{bmatrix} a_1 & 0 & a_3 & 0 \\ a_5 & 0 & a_7 & 0 \\ a_9 & 0 & a_{11} & 0 \end{bmatrix};$$

W_1 is a S-special linear operator on V.

Several types of linear operators can be defined on V.

For instance U : V → V defined by

$$U (\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix}) = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix}$$

is a S-special linear operator on V.

$$\text{Clearly kernel } U \neq \{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\}.$$

Thus we can have several types of linear operators.

Example 3.24: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{bmatrix} \right| a_i \in [0, 58), 1 \leq i \leq 25\}$$

be the S-special interval vector space over the S-ring R = Z_{58}.

Let T : V → V be defined by

$$T(\begin{bmatrix} a_1 & \dots & a_5 \\ a_6 & \dots & a_{10} \\ \vdots & & \vdots \\ a_{21} & \dots & a_{25} \end{bmatrix}) = \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_5 & 0 \\ 0 & 0 & 0 & 0 & a_6 \end{bmatrix}$$

is a S-special linear operator on V.

Define T_1 : V → V be

$$T_1(\begin{bmatrix} a_1 & \dots & a_5 \\ a_6 & \dots & a_{10} \\ \vdots & & \vdots \\ a_{21} & \dots & a_{25} \end{bmatrix}) = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

T_1 is S-special linear operator on V.

$$\text{Clearly ker } T_1 \neq \begin{bmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix}.$$

Define $T_2 : V \to V$ is defined by

$$T_2 \left(\begin{bmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ \vdots & \vdots & & \vdots \\ a_{21} & a_{22} & \dots & a_{25} \end{bmatrix} \right) = \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 \\ a_2 & a_3 & 0 & 0 & 0 \\ a_4 & a_5 & a_6 & 0 & 0 \\ a_7 & a_8 & a_9 & a_{10} & 0 \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{bmatrix};$$

T is a S-special linear operator on V.

$$\ker T_2 \neq \begin{bmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix}.$$

$T_3 : V \to V$ is defined by

$$T_3 \left(\begin{bmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ \vdots & \vdots & & \vdots \\ a_{21} & a_{22} & \dots & a_{25} \end{bmatrix} \right) = \begin{bmatrix} 0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & a_5 & a_6 & a_7 \\ 0 & 0 & 0 & a_8 & a_9 \\ 0 & 0 & 0 & 0 & a_{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is again a S-special linear operator on V and

$$\ker T_3 \neq \begin{bmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix}.$$

Next we proceed onto derive other properties associated with these S-special type of vector spaces.

We can also define the notion of S-special interval pseudo linear algebra over Z_n ($n < \infty$) and Z_n a S-ring.

In the first place S-special interval vector space V over a S-ring; $R = Z_n$ is defined as the S-special interval pseudo linear algebra if V is endowed with a product '.' such that for $a, b \in V$. $a . b \in V$ and '.' is associative on V and for $a \in R$, $v_1, v_2 \in V$.

$a (v_1 + v_2) \neq av_1 + av_2 \in V$ in general the distributive law may or may not be true.

We will illustrate this situation by some examples.

Example 3.25: Let $V = \{[0, 22), +, \times\}$ be the S-special pseudo interval linear algebra over the S-ring $R = Z_{22}$.

If $x = 11.5$ and $y = 5 \in V$. $x . y = 11.5 \times 5 = 57.5 = 13.5 \in V$.

This is the way V is a S-special pseudo interval linear algebra over the S-ring Z_{22}.

We see V is an infinite dimensional over the S-ring $R = Z_{22}$. We see V has S-special interval pseudo linear subalgebras over $R = Z_{22}$ of finite order over R.

$M = \{Z_{22}, +, \times\}$, the S-special interval pseudo linear subalgebra over R is of dimension one over $R = Z_{22}$.

$P = \{\{0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, \ldots, 20, 20.5, 21, 21.5\}\} \subseteq V$ is a S-special interval pseudo linear subalgebra of finite dimension over $R = Z_{22}$.

Example 3.26: Let $V = \{[0, 12), +, \times\}$ be the S-special pseudo interval linear algebra over the S-ring= Z_{12}. V has finite dimensional sublinear pseudo algebras over the S-ring $R = Z_{12}$.

$M_1 = \{Z_{12}, \times, +\} \subseteq V$ is a S-special interval pseudo sublinear algebra of V over the S-ring of dimension 1 over $R = Z_{12}$.

$M_2 = \{\{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, \ldots, 11, 11.2, 11.4, 11.6, 11.8\} \subseteq [0, 12)\} \subseteq V$ is a S-special quasi vector subspace of finite dimension over the S-ring $R = Z_{12}$.

Let $W = \{\{0, 0.1, 0.2, \ldots, 0.9, 1, 1.1, 1.2, \ldots, 11.9\} \subseteq [0, 12)\} \subseteq V$ be a finite dimensional quasi vector subspace of V over $R = Z_{12}$.

However $T_1 = \{\{0, 4, 8\} \subseteq Z_{12} \subseteq [0, 12)\}$ is a finite dimensional S-linear subalgebra of V over Z_{12}.

Likewise $T_2 = \{\{0, 2, 4, 6, 8, 10\} \subseteq Z_{12} \subseteq [0, 12)\} \subseteq V$ is again a S-special pseudo sublinear algebra of V over $R = Z_{12}$.

$T_3 = \{\{0, 3, 6, 9\} \subseteq Z_{12} \subseteq [0, 12)\} \subseteq V$ is also a finite dimensional S-special pseudo linear subalgebra of V over $R = Z_{12}$.

V has finite dimensional S-special linear pseudo subalgebras as well as S-special quasi vector subspaces over the S-ring Z_{12}.

Example 3.27: Let $V = \{[0, 46), +, \times\}$ be the S-special pseudo interval linear algebra over the S-ring Z_{46}.

V has finite dimensional S-special linear subalgebras of finite order though V is an S-infinite dimensional linear algebra over the S-ring Z_{46}.

Further $P_1 = \{Z_{46}, +, \times\} \subseteq V$ is a one dimensional S-special linear subalgebra of dimension one over the S-ring Z_{46}.

We see $M_1 = \{\{0, 0.5, 1, 1.5, 2, \ldots, 44, 44.5, 45, 45.55\} \subseteq [0, 46)\} \subseteq V$ is not a S-special linear pseudo subalgebra only a S-special quasi vector subspace of V over $R = Z_{46}$.

However M_1 is a finite dimensional S-quasi vector subspace of V over the S-ring $R = Z_{46}$.

$M_2 = \{\{0, 0.1, 0.2, 0.3, \ldots, 0.9, 1, 1.1, 1.2, \ldots, 44, 44.1, \ldots, 44.9, 45, 45.1, \ldots, 45.9\} \subseteq [0, 46)\} \subseteq V$ is a S-quasi vector subspace of V over the S-ring $R = Z_{46}$.

M_2 is a finite dimensional S-quasi vector subspace over the S-ring. $M_3 = \{\{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, \ldots, 1.8, 2, 2.2, \ldots, 45, 45.2, \ldots, 45.4, 45.6, 45.8\} \subseteq [0, 46)\}$ is a S-special quasi vector subspace of V over the S-ring Z_{46}.

Clearly M_3 is also finite dimensional over $R = Z_{46}$. $T_1 = \{\{0, 23\} \subseteq Z_{46} \subseteq [0, 46)\} \subseteq V$ is again a S-special sublinear algebra of V over the S-ring Z_{46} and

$T_2 = \{\{0, 2, 4, 6, \ldots, 44\} \subseteq [0, 46)\} \subseteq V$ is again a S-special sublinear algebra of V over the S-ring Z_{46}, both T_1 and T_2 are both finite dimensional S-special pseudo linear subalgebra of V over the S-ring $R = Z_{46}$.

Example 3.28: Let $V = \{[0, 93),. +, \times\}$ be the S-special pseudo interval linear algebra over the S-ring $R = Z_{93}$.

This S-special interval pseudo linear algebras has both S-linear subalgebras as well as S-quasi vector subspaces which are finite dimensional over R.

Now we proceed onto give S-special pseudo linear algebras built using the interval [0, n) where Z_n is a S-ring.

Example 3.29 : Let $V = \{(a_1, a_2, a_3, a_4) \mid a_i \in [0, 42), 1 \leq i \leq 4\}$ be a S-special pseudo linear algebra over the S-ring $R = Z_{42}$. V has both finite and infinite dimensional linear subalgebras over $R = Z_{42}$.

Further V can be written as a direct sum of S-sublinear algebras.

Let $W_1 = \{(a_1, 0, 0, 0) \mid a_1 \in [0, 42), +, \times\} \subseteq V$ be a S-special interval pseudo sublinear algebra of infinite dimension over $R = Z_{42}$.

Let $W_2 = \{(0, a_2, 0, 0) \mid a_2 \in [0, 42), +, \times\} \subseteq V$,
$W_3 = \{(0, 0, a_3, 0) \mid a_3 \in [0, 42), +, \times\} \subseteq V$ and
$W_4 = \{(0, 0, 0, a_4) \mid a_4 \in [0, 42), +, \times\} \subseteq V$ be the four S-special interval pseudo linear subalgebras of V over $R = Z_{42}$.

Clearly $W_i \cap W_j = (0, 0, 0, 0)$ if $i \neq j$, $1 \leq i, j \leq 4$ and $V = W_1 + W_2 + W_3 + W_4$ is a direct sum of sublinear pseudo algebras over the S-ring $R = Z_{42}$.

Let $P_1 = \{(a_1, 0, 0, 0) \mid a_1 \in Z_{42}, +, \times\} \subseteq V$,
$P_2 = \{(0, a_2, 0, 0) \mid a_2 \in Z_{42}, +, \times\} \subseteq V$,
$P_3 = \{(0, 0, a_3, 0) \mid a_3 \in Z_{42}, +, \times\} \subseteq V$ and
$P_4 = \{(0, 0, 0, a_4) \mid a_4 \in Z_{42}, +, \times\} \subseteq V$ are the four S-special interval pseudo linear subalgebras of V over $R = Z_{42}$.

Clearly $P_i \cap P_j = \{(0, 0, 0, 0)\}$ if $i \neq j$, $1 \leq i, j \leq 4$ and $P = P_1 + P_2 + P_3 + P_4 \not\subseteq V$ is also a finite dimensional linear subalgebra of V over $R = Z_{42}$. Thus this sort of direct sum we define as sub subdirect sum of sublinear algebras of V.

Let $M_1 = \{(a_1, a_2, 0, 0) \mid a_1, a_2 \in \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 41, 41.5\}, +, \times\} \subseteq V$ be a S-quasi special vector subspace of V over the S-ring Z_{42}.

M_1 is finite dimensional; M_1 is not a linear pseudo subalgebra for if $x = (0.5, 2.5, 0, 0)$ and $y = (1.5, 2.5, 0, 0) \in M$, then $x \times y = (0.5, 2.5, 0, 0) \times (1.5, 2.5, 0, 0) = (7.25, 6.25, 0, 0) \notin M_1$.

Hence the claim.

Let $N_1 = \{(a_1, 0, a_2, 0) \mid a_1 \in [0, 42)$ $a_2 \in \{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, \ldots, 40, 40.2, \ldots, 40.8, 41, 41.2, 41.4, 41.6, 41.8\} \subseteq$

[0. 42), +, ×} $\subseteq$ V be the only S-special interval quasi vector subspace of V over R = Z_{42}.

Infact N_1 is an infinite dimensional S-special quasi interval vector subspace of V over R = Z_{42}.

Cleary N_1 is a S-special interval pseudo linear subalgebra of V over the S-ring Z_{42}.

Let x = (5, 0, 0.4, 0) and y = (3, 0, 2.4, 0) $\in N_1$;

$$\begin{aligned} x \times y &= (5, 0, 0.4, 0) \times (3, 0, 2.4, 0) \\ &= (15, 0, 0.96, 0) \notin N_1. \end{aligned}$$

Hence the claim.

Thus we have S-special quasi subset vector subspaces of V of both finite and infinite dimension over the S-ring R = Z_{42}.

Let $S_1 = \{(a_1, a_2, 0, 0) \mid a_1, a_2 \in Z_{42}\} \subseteq V$ be a S-special interval sublinear algebra of finite dimension over R = Z_{42}. $S_2 = \{(a_1, a_2, 0, 0) \mid a_1, a_2 \in \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 41, 41.5\} \subseteq [0, 42)\} \subseteq V$ is only a S-special quasi vector subspace of V over the S-ring R = Z_{42}.

Clearly S_2 is not a S-special linear subalgebra of V for if x = (0.5, 1.5, 0, 0) and y = (0.5, 0.5, 0, 0) $\in S_2$ then
x + y = (0.5, 1.5, 0, 0) + (0.5, 0.5, 0, 0) = (1, 2, 0, 0) $\in S_2$ but
x × y = (0.5, 1.5, 0, 0) × (0.5, 0.5, 0, 0) = (0.25, 0.75, 0, 0) $\notin S_2$.

So S_2 is not closed under the product hence S_2 is only a S-special quasi vector subspace of V over R = Z_{42}.

However dimension of S_2 over the ring R is finite.

Let $S_3 = \{(0, a_1, 0, a_2) \mid a_1, a_2 \in \{0, 0.1, 0.2, \ldots, 0.9, 1, 1.1, \ldots, 1.9, \ldots, 41.1, 41.2, \ldots, 41.9\} \subseteq [0, 42)\} \subseteq V$ be the only finite dimensional S-special quasi vector subspace of V and is not a S-special pseudo linear subalgebra of V over the S-ring R = Z_{42}.

We have seen both finite and infinite dimensional S-special quasi vector subspaces of V. Now consider $S_4 = \{(a_1, a_2, a_3, a_4) \mid a_i \in \{0\ 2, 4, 6, 8, \ldots, 40\} \subseteq Z_{42} \subseteq [0, 42); 1 \leq i \leq 4\} \subseteq V$ is a S-special linear subalgebra of V over the S-ring Z_{42}. Clearly dimension of S_4 over Z_{42} is finite.

We have only finite number of S-special linear subalgebras of V over the S-ring $R = Z_{42}$.

Example 3.30: Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \end{bmatrix} \middle| \; a_i \in [0, 6); 1 \leq i \leq 6 \right\}$$

be the S-special vector space over the S-ring $R = Z_6$.

V is infinite dimensional over the S-ring $R = Z_6$. V has several infinite dimensional S-special vector subspaces over Z_6.

For

$$P_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \; a_1 \in [0, 6) \right\} \subseteq V$$

is a S-special vector subspace of V over the S-ring $R = Z_6$.

$$P_2 = \left\{ \begin{bmatrix} p_1 \\ p_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \; p_1, p_2 \in [0, 6) \right\} \subseteq V$$

is a S-special vector subspace of V over the S-ring R = Z_6 of infinite dimension over Z_6.

$$P_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_1 \in [0, 6) \right\} \subseteq V$$

is a S-special vector subspace of V over the S-ring R = Z_6.

$$P_4 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \middle| \; a_i \in [0, 6), 1 \leq i \leq 3 \right\} \subseteq V$$

is a S-special vector subspace of V over the S-ring R = Z_6.

$$P_5 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ a_2 \\ 0 \\ a_3 \\ 0 \end{bmatrix} \middle| \; a_i \in [0, 6), 1 \leq i \leq 3 \right\} \subseteq V$$

is a S-special vector subspace of V over the S-ring R = Z_6.

Thus V has several S-special vector subspace of infinite dimension over the S-ring Z_6.

Consider

$$S_1 = \left\{ \left. \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right| a_1 \in Z_6 \right\} \subseteq V$$

is a S-special vector subspace of V over the S-ring Z_6 and is finite dimensional over Z_6.

Let

$$S_2 = \left\{ \left. \begin{bmatrix} 0 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right| a_2 \in Z_6 \right\} \subseteq V$$

is a S-special vector subspace of V over the S-ring Z_6 and is finite dimensional over Z_6.

$$S_3 = \left\{ \left. \begin{bmatrix} a_1 \\ 0 \\ a_2 \\ 0 \\ a_3 \\ 0 \end{bmatrix} \right| a_i \in Z_6, 1 \leq i \leq 3 \right\} \subseteq V$$

is a S-special vector subspace of V over Z_6 of finite dimension.

$$S_4 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \end{bmatrix} \middle| \; a_1 \; a_2 \in Z_6 \subseteq [0, 6) \right\} \subseteq V$$

is a S-special vector subspace of finite dimension over the S-ring Z_6.

$$S_5 = \left\{ \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_1 \end{bmatrix} \middle| \; a_1 \in Z_6 \right\} \subseteq V$$

is a finite dimensional S-special vector subspace of V over the S-ring Z_6.

We can make V into a S-special pseudo linear algebra by defining a product $\times_n$, the natural product on V.

Thus (V, $\times_n$) becomes a S-special interval pseudo linear algebra of infinite dimension over Z_6.

However V has also finite dimensional S-special linear subalgebras over Z_6.

Let

$$M_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \; a_1 \in Z_6, \times_n \right\} \subseteq V$$

be a S-special linear subalgebra of V over the S-ring Z_6. M_1 is finite dimensional over Z_6.

$$M_2 = \left\{ \left. \begin{bmatrix} 0 \\ a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right| a_1, a_2 \in Z_6, \times_n \right\} \subseteq V$$

is a S-special linear subalgebra of V over the S-ring Z_6.

M_2 is of dimension two over Z_6.

The basis of M_2 is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \subseteq M_2$ over Z_6.

$$M_3 = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \end{bmatrix} \right| a_i \in Z_6, 1 \leq i \leq 6 \right\} \subseteq V$$

is a S-special pseudo linear subalgebra of V over the S-ring Z_6.

Dimension of M_3 over the S-ring Z_6 is 6. The basis of M_3 is given by

$$B = \left\{ \begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix} \right\} \subseteq M_3 \text{ over } Z_6.$$

$$M_4 = \left\{ \begin{bmatrix}a_1\\a_2\\\vdots\\a_6\end{bmatrix} \middle| \, a_i \in \{0, 2, 4\} \subseteq Z_6, 1 \leq i \leq 6 \right\} \subseteq V$$

is a S-special linear subalgebra of V over Z_6.

Clearly M_4 is of dimension 6 over Z_6.

$$\text{A basis of } M_4 \text{ is } \left\{ \begin{bmatrix}2\\0\\0\\0\\0\\0\end{bmatrix} \begin{bmatrix}0\\2\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\2\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\2\end{bmatrix} \right\} \subseteq V.$$

$$M_5 = M_4 = \left\{ \begin{bmatrix}a_1\\a_2\\a_3\\0\\0\\0\end{bmatrix} \middle| \, a_i \in \{0, 3\}, 1 \leq i \leq 3 \right\} \subseteq V$$

is a S-special linear subalgebra of V over the S-ring Z_6. M_5 is finite dimensional over Z_6.

$$\text{A basis of } M_5 \text{ over V is } \left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Thus V has S-special sublinear algebras of finite dimension as well as infinite dimension over Z_6.

$$T_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ a_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_1, a_2 \in \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 4.5, 5, \right.$$

$$\left. 5.5\} \subseteq [0, 6) \right\} \subseteq V$$

is only a S-special quasi vector subspace of V of finite dimension over Z_6.

Clearly T_2 is not a S-special linear subalgebra of V over Z_6.

$$\text{For take } x = \begin{bmatrix} 0.5 \\ 0 \\ 1.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 1.5 \\ 0 \\ 2.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in T_1.$$

We see $x \times_n y = \begin{bmatrix} 0.5 \\ 0 \\ 1.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} \times_n \begin{bmatrix} 1.5 \\ 0 \\ 2.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0 \\ 3.75 \\ 0 \\ 0 \\ 0 \end{bmatrix} \notin T_2$;

hence T_2 is not a S-special linear subalgebra only a S-special quasi vector subspace of V over R = Z_6 is of finite dimension.

$$\text{Let } T_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \end{bmatrix} \middle| \; a_1, a_2 \in \{0, 0.01, 0.02, \ldots, 0.1, \ldots, 1.0, 1.01, \ldots, 5.01, 5.,02, \ldots, 5.99\} \subseteq [0, 6), +\} \subseteq V\right.$$

be a S-special quasi vector subspace of V over the S-ring Z_6.

Clearly T_3 is a S-special linear subalgebra of V over Z_6.

$$\text{Let } T_4 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_2 \end{bmatrix} \middle| \; a_1, a_2 \in \{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, \ldots, 5, 5.2, 5.4, 5.6, 5.8\} \subseteq [0, 6)\} \subseteq V\right.$$

be a S-special quasi vector subspace of V over the S-ring Z_6 and is not a S-special linear subalgerba of V over Z_6.

Clearly T_4 is finite dimensional over Z_6.

Let $T_5 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \right.$ $a_1 \in \{0, 0.25, 0.50, 0.75, 1, 1.25, \ldots, 5, 5.25, 5.5, 5.75\} \subseteq [0, 6)$,

and $a_2 \in \{0, 0.001, 0.002, \ldots, 1, 1.001, 1.002, \ldots, 4.001, \ldots, 4.999\} \subseteq [0, 6)\} \subseteq V$ be a S-special interval quasi vector subspace of V over the S-ring Z_6.

Clearly dimension of T_5 over Z_6 is finite dimensional.

We can write $V = W_1 + \ldots + W_6$ or
$V = W_1 + W_2$ or
$V = W_1 + W_2 + W_3$ or
$V = W_1 + W_2 + W_3 + W_4$ and
$V = W_1 + W_2 + W_3 + W_4 + W_5$.

We just show this by some illustration.

$$\text{Let } W_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \; a_1 \in [0, 6), \times_n \right\} \subseteq V,$$

$$W_2 = \left\{ \left. \begin{bmatrix} 0 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right| a_2 \in [0, 6), \times_n \right\} \subseteq V,$$

$$W_3 = \left\{ \left. \begin{bmatrix} 0 \\ 0 \\ a_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right| a_3 \in [0, 6), \times_n \right\} \subseteq V,$$

$$W_4 = \left\{ \left. \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \\ 0 \\ 0 \end{bmatrix} \right| a_4 \in [0, 6), \times_n \right\} \subseteq V,$$

$$W_5 = \left\{ \left. \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_5 \\ 0 \end{bmatrix} \right| a_5 \in [0, 6), \times_n \right\} \subseteq V \text{ and}$$

$$W_6 = \left\{ \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_6 \end{array} \right] \right| a_6 \in [0, 6), \times_n \} \subseteq V$$

be the six S-special pseudo linear subalgebras of V over the S-ring Z_6.

Clearly

$$W_i \cap W_j = \left\{ \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right] \right\} \text{ if } i \neq j,\ 1 \leq i, j \leq 6$$

and $V = W_1 + \ldots + W_6$ and thus V is the direct sum of S-special linear subalgebras over Z_6.

Let

$$B_1 = \left\{ \left[\begin{array}{c} a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right] \right| a_1, a_2 \in [0, 6), \times_n \} \subseteq V,$$

$$B_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ a_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \, a_3 \in [0, 6), \times_n \right\} \subseteq V,$$

$$B_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \\ 0 \\ 0 \end{bmatrix} \middle| \, a_4 \in [0, 6), \times_n \right\} \subseteq V,$$

$$B_4 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_5 \\ 0 \end{bmatrix} \middle| \, a_5 \in [0, 6), \times_n \right\} \subseteq V,$$

$$B_5 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_6 \end{bmatrix} \middle| \, a_6 \in [0, 6), \times_n \right\} \subseteq V$$

be the S-special linear subalgebras of V over the S-ring Z_6.

We see

$$B_i \cap B_j = \left\{\begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix}\right\} \text{ if } i \neq j,\ 1 \leq i, j \leq 6 \text{ and}$$

V = $B_1 + B_2 + B_3 + B_4 + B_5$ is the direct sum of S-special sublinear algebras of V.

Let

$$C_1 = \left\{\begin{bmatrix}a_1\\0\\a_2\\0\\a_3\\0\end{bmatrix} \middle|\ a_i \in [0, 6),\ 1 \leq i \leq 3, \times_n\right\} \subseteq V \text{ and}$$

$$C_2 = \left\{\begin{bmatrix}0\\a_1\\0\\a_2\\0\\a_3\end{bmatrix} \middle|\ a_i \in [0, 6),\ 1 \leq i \leq 3, \times_n\right\} \subseteq V$$

be the two S-special linear subalgebras of V over Z_6.

We have

$$C_1 \cap C_2 = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\} \text{ and } V = C_1 + C_2$$

is the direct sum of S-sublinear algebras of V.

$$\text{Let } D_1 = \left\{\begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_2 \end{bmatrix} \middle| \; a_1, a_2 \in [0, 6), \times_n\right\} \subseteq V,$$

$$D_2 = \left\{\begin{bmatrix} 0 \\ a_1 \\ 0 \\ 0 \\ a_2 \\ 0 \end{bmatrix} \middle| \; a_1, a_2 \in [0, 6), \times_n\right\} \subseteq V \text{ and}$$

$$D_3 = \left\{\begin{bmatrix} 0 \\ 0 \\ a_1 \\ a_2 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_1, a_2 \in [0, 6), \times_n\right\} \subseteq V$$

be S-special linear subalgebras of V over the S-ring Z_6.

Clearly

$$D_i \cap D_j = \left\{\begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix}\right\}, i \neq j, 1 \leq i, j \leq 3 \text{ and}$$

$D_1 + D_2 + D_3 = V$ is the direct sum of S-sublinear algebras. Let

$$E_1 = \left\{\begin{bmatrix}a_1\\a_2\\0\\0\\0\\0\end{bmatrix}\middle| a_1, a_2 \in Z_6\right\} \subseteq V,$$

$$E_2 = \left\{\begin{bmatrix}0\\0\\0\\a_1\\a_2\\a_3\end{bmatrix}\middle| a_1, a_2, a_3 \in Z_6\right\} \subseteq V \text{ and}$$

$$E_3 = \left\{\begin{bmatrix}0\\0\\0\\0\\0\\a_1\end{bmatrix}\middle| a_1 \in Z_6\right\} \subseteq V$$

be the three S-special linear subalgebras of V.

$$E_i \cap E_j = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}, \; i \neq j, \; 1 \leq i, j \leq 3.$$

But $V \neq E_1 + E_2 + E_3$.

$$\text{Further } E = E_1 + E_2 + E_3 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| \; a_i \in Z_6, \; 1 \leq i \leq 6 \right\} \subseteq V$$

is a S-special pseudo linear subalgebra of finite dimension over $R = Z_6$.

Now we give some more examples before we proceed onto discuss about other properties.

Example 3.31: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \vdots & \vdots & \vdots & \vdots \\ a_{49} & a_{50} & a_{51} & a_{52} \end{bmatrix} \middle| \; a_i \in [0, 46), \; 1 \leq i \leq 52 \right\}$$

be the S- special vector space over the S-ring $R = Z_{46}$. If the natural product $\times_n$ on matrices is defined V becomes a S-special pseudo linear algebra over the S-ring $R = Z_{46}$.

It is easily verified V can be written as a direct sum. V has both finite and infinite dimensional S-special linear subalgebras. Also V has both finite and infinite dimensional S-special quasi vector subspaces over Z_{46}.

Let

$$W_1 = \left\{ \left[\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{array} \right] \right| a_i \in \{0, 0.5, 1, 1.5, 2, \ldots, 40,$$

$$40.5, 41, 41.5, 42, 42.5, \ldots, 45, 45.5\} 1 \le i \le 4\} \subseteq V$$

be a S-special quasi vector subspace of V and is not a S-special linear subalgebra of V.

Clearly dimension of $W_1 \subseteq V$ over $R = Z_{46}$ is finite dimensional.

Let

$$W_2 = \left\{ \left[\begin{array}{cccc} a_1 & 0 & 0 & a_2 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{array} \right] \right| a_i \in \{0, 0.2, 0.4, 0.6, 0.8, 1, \ldots,$$

$$45, 45.2, \ldots, 45.8\} \text{ and } a_2 \in [0, 46)\} \subseteq V.$$

W_2 is a S-special quasi vector subspace of V over $R = Z_{46}$. The dimension of W_2 over Z_{46} is infinite.

However all the S-special interval pseudo linear algebras defined over the S-ring in this chapter are commutative, now we proceed onto give examples of non commutative S-special interval pseudo linear algebras over the S-ring.

Example 3.32: Let
$V = \{(a_1 \mid a_2\ a_3\ a_4 \mid a_5) \mid a_i \in [0, 58), 1 \leq i \leq 5, +, \times\}$ be the S-special interval pseudo linear algebra over the S-ring Z_{58}.

V has both finite and infinite dimensional S-special linear subalgebras. Further V has both finite and infinite dimensional S-special quasi vector subspaces.

Take $P_1 = \{(a_1 \mid a_2\ a_3\ a_4 \mid a_5) \mid a_i \in Z_{58}, 1 \leq i \leq 5\} \subseteq V$, V is a finite dimensional S-special linear subalgebra of V over the S-ring $R = Z_{58}$.

Let $P_2 = \{(a_1 \mid a_2\ 0\ 0 \mid a_3) \mid a_i \in [0, 58), 1 \leq i \leq 58\} \subseteq V$; P_2 be a infinite dimensional S-special linear subalgebra of V over the S-ring Z_{58}.

Let $P_3 = \{(a_1 \mid 0\ 0\ a_2 \mid 0) \mid a_1, a_2 \in \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 57, 57.5\} \subseteq [0, 58)\} \subseteq V$ be a S-special quasi vector subspace of V over S-ring Z_{58}. P_3 is finite dimensional over S-ring Z_{58}.

$P_4 = \{(0 \mid a_1\ a_2\ a_3 \mid 0) \mid a_1, a_2 \in [0, 58), a_3 \in \{0, 0.1, 0.2, \ldots, 0.9, 1, 1.1, \ldots, 57.1, 57.2, \ldots, 57.9\} \subseteq 0.58)\} \subseteq V$ is an infinite dimensional S-special quasi vector subspace of V over Z_{58}.

Example 3.33: Let

$$V = \left\{ \left[\begin{array}{c} a_1 \\ \hline a_2 \\ a_3 \\ \hline a_4 \\ a_5 \\ a_6 \\ \hline a_7 \\ a_8 \end{array} \right] \middle|\ a_i \in [0, 51), 1 \leq i \leq 8, +, \times_n \right\}$$

be a S-special pseudo interval linear algebra over the S-ring $R = Z_{51}$ under the natural product of matrices.

We have finite and infinite dimensional S-special pseudo linear subalgebras as well as finite and infinite dimensional S-special quasi vector subspaces.

$$\text{Let } T_1 = \left\{ \left[\begin{array}{c} 0 \\ \hline a_1 \\ a_2 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \end{array} \right] \middle| \; a_1, a_2 \in [0, 51), +, \times_n \right\} \subseteq V \text{ and}$$

$$T_2 = \left\{ \left[\begin{array}{c} 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline a_1 \\ a_2 \end{array} \right] \middle| \; a_1, a_2 \in [0, 51), +, \times_n \right\} \subseteq V$$

be two S-special pseudo interval linear subalgebra of V.

We see both T_1 and T_2 are infinite dimensional S-special interval linear subalgebras of V over the S-ring Z_{51}.

It is observed that for every $x \in T_1$ and every $y \in T_2$ are such that

$$x \times_n y = \left[\begin{array}{c} 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right].$$

Thus the S-special linear subalgebra T_1 is orthogonal to the S-special linear subalgebra T_2, however T_1 is not the orthogonal complement of T_2 in V.

Now consider

$$T_3 = \left\{ \left[\begin{array}{c} 0 \\ \hline 0 \\ 0 \\ \hline a_1 \\ a_2 \\ a_3 \\ \hline 0 \\ 0 \end{array}\right] \middle| \, a_1, a_2, a_3 \in [0, 51), +, \times_n \right\} \subseteq V$$

is also a S-special interval pseudo linear subalgebra of V over the S-ring Z_{51}.

T_3 is also infinite dimensional; T_3 is orthogonal to both T_1 and T_2 however T_3 is not the orthogonal complement of T_1 or T_2.

$$\text{Let } T_4 = \left\{ \left[\begin{array}{c} a_1 \\ \hline a_2 \\ a_3 \\ \hline 0 \\ 0 \\ 0 \\ \hline a_4 \\ a_5 \end{array} \right] \middle| \; a_i \in [0, 51), 1 \leq i \leq 5, +, \times_n \right\} \subseteq V$$

be a S-special interval pseudo linear subalgebra of V over the S-ring Z_{51}.

We see the orthogonal complement of T_4 is T_3 and vice versa.

Further $V = T_3 + T_4$.

We can also discuss as in case of usual linear algebras notion of orthogonal complement of a set $S \subseteq V$.

However it is left as an exercise to find $S^{\perp} = \{x \in V \mid x \times_n y = (0) \text{ for all } y \in S\}$ and prove $S^{\perp}$ is a S-special vector subspace of V.

Example 3.34: Let

$$V = \left\{ \left(\begin{array}{c|cc|ccc|cc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \end{array} \right) \middle| \; a_i \in [0, 62), \right.$$

$$\left. 1 \leq i \leq 24 \right\}$$

be the S-special interval pseudo linear algebra under the natural product $\times_n$ over the S-ring Z_{62}.

We see

$$M_1 = \left\{ \left(\begin{array}{c|cc|ccc|cc} a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \middle| \; a_i \in [0, 62), \right.$$

$$\left. 1 \leq i \leq 3 \right\} \subseteq V$$

is a S-special interval pseudo linear subalgebra of V over Z_{62}.

$$M_2 = \left\{ \left(\begin{array}{c|cc|ccc|cc} 0 & a_1 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_3 & a_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_5 & a_6 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \middle| \; a_i \in [0, 62), \right.$$

$$\left. 1 \leq i \leq 6 \right\} \subseteq V$$

is again a S-special interval pseudo linear subalgebra of V over the S-ring Z_{62}.

Now

$$M_3 = \left\{ \left(\begin{array}{c|cc|ccc|cc} 0 & 0 & 0 & a_1 & 0 & a_4 & 0 & 0 \\ 0 & 0 & 0 & a_2 & 0 & a_5 & 0 & 0 \\ 0 & 0 & 0 & a_3 & 0 & a_6 & 0 & 0 \end{array} \right) \middle| \; a_i \in [0, 62), \right.$$

$$\left. 1 \leq i \leq 6 \right\} \subseteq V$$

is a S-special interval sublinear algebra of V over the S-ring Z_{62}.

$$\text{Clearly } M_i \cap M_j = \left\{ \left(\begin{array}{c|cc|ccc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \right\} \text{ if } i \neq j,$$

$$1 \leq i, j \leq 3$$

and for every $x \in M_j$ and for every

$$y \in M_i \; (i \neq j; \; x \times_n y = \left\{\left(\begin{array}{c|cc|ccc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)\right\},$$

$1 \leq i, j \leq 3.$

Let $N_1 = \left\{\left(\begin{array}{c|cc|ccc|cc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \right|$ $a_i \in \{0, 0.5, 1,$

$1.5, 2, 2.5, \ldots, 60, 60.5, 61, 61.5\} \subseteq [0, 62), 1 \leq i \leq 8\} \leq V;$

N_1 is a S-special interval quasi vector subspace of V. N_1 is a S-special interval quasi vector subspace of V.

N_1 is not closed under product $\times_n$. N_1 is a finite dimensional S-special quasi vector subspace of V over the S-ring Z_{62}.

Let

$N_2 = \left\{\left(\begin{array}{c|cc|ccc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & a_5 & a_6 & a_7 & a_8 \end{array}\right) \right|$ $a_i \in \{0, 0.1, 0.2,$

$0.3, \ldots, 0.9, 1, 1.1, 1.2, \ldots, 60, 1, 60.2, \ldots, 60.9, 61,$
$61.1, \ldots, 61.9\} \subseteq [0, 62), 1 \leq i \leq 8\} \subseteq V,$

N_2 is also a S-special interval quasi vector subspace of V over the S-ring Z_{62} and N_2 is of finite dimension over Z_{62}. However N_1 is orthogonal with N_2 and vice versa.

But N_1 is not the orthogonal complement of N_2 and vice versa.

Infact $N^{\perp}$ of N_1 is a S-special interval subspace of V over Z_{62}.

$$N^{\perp} = \left\{ \left(\begin{array}{c|cc|ccc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \end{array} \right) \middle| \; a_i, a_j \in [0, 62), 1 \leq i, j \leq 16 \right\} \subseteq V$$

is S-special pseudo linear subalgebra of V over Z_{62}.

$$\text{We see } N_1 \cap N^{\perp} = \left\{ \left(\begin{array}{c|cc|ccc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \right\}$$

$$\text{and } N_1 \times_n N^{\perp} = \left\{ \left(\begin{array}{c|cc|ccc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \right\}$$

but however $N_1 + N^{\perp} \neq V$. $N^{\perp}$ is the orthogonal complement of N_1.

Example 3.35: Let

$$V = \left\{ \left[\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ \hline a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{25} & a_{26} & a_{27} & a_{28} \\ a_{29} & a_{30} & a_{31} & a_{32} \\ a_{33} & a_{34} & a_{35} & a_{36} \\ a_{37} & a_{38} & a_{39} & a_{40} \end{array} \right] \middle| \; a_i \in [0, 69), 1 \leq i \leq 40 \right\}$$

be the S-special pseudo interval linear algebra over the S-ring Z_{69}.

V has infinite and finite dimensional S-interval quasi vector subspace over the S-ring Z_{69}.

$$\text{Let } M_1 = \left\{ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \right| a_i \in \{0, 0.5, 1, 1.5, 2, 2.5, 3, \ldots,$$

$$68, 68{,}5\} \subseteq [0, 69), 1 \leq i \leq 4\} \subseteq V$$

be the S-special quasi vector subspace of V over the S-ring Z_{69}. Clearly M_1 is a finite dimensional S-special quasi vector subspace of V.

$$\text{Let } M_2 = \left\{ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \right| a_1, a_2, a_3, a_4 \in [0, 69), a_5, a_6,$$

$$a_7, a_8 \in \{0, 0.1, 0.2, 0.3, \ldots, 0.9, 1, 1.1, 1.2, \ldots, 1.9, 2, \ldots, 68, 68.1, 68.2, \ldots, 68.9\} \subseteq [0, 69) \subseteq V$$

be a S-special quasi vector subspace of V over the S-ring Z_{69}.

However M_2 is a S-special quasi vector subspace of infinite dimensional over the S-ring Z_{69}. It is easily verified M_1 is orthogonal to M_2 and vice versa.

But M_1 is not the orthogonal complement of M_2 and M_2 is not orthogonal complement of M_1.

Let

$$N_1 = \left\{ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline a_1 & a_2 & a_3 & a_4 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \middle| \; a_i \in [0, 69); 1 \leq i \leq 64 \right\} \subseteq V$$

be the S-special pseudo interval linear subalgebra of V over the S-ring Z_{69}.

The dimension of N_1 is infinite over the S-ring Z_{69}.

$$\text{Let } N_2 = \left\{ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \\ a_5 & a_6 & a_7 & a_8 \end{array} \right] \middle| \; a_i \in [0, 69); 1 \leq i \leq 8 \right\} \subseteq V$$

be the S-special pseudo interval linear subalgebra of V over the S-ring Z_{69}.

We see N_1 is orthogonal with N_2 and N_2 is orthogonal with N_1. N_1 is not the orthogonal complement of N_2 and N_2 is not the orthogonal complement of N_1 and $N_1 + N_2 \neq V$.

Example 3.36: Let

$$V = \left\{ \left[\begin{array}{c|cc|c} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \hline a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} \end{array} \right] \middle| \; a_i \in [0, 55), 1 \leq i \leq 24 \right\}$$

be the S-special interval pseudo linear algebra over the S-ring Z_{55}.

V has finite and infinite dimensional S-special quasi vector subspaces.

$$M_1 = \left\{\left[\begin{array}{c|cc|c} a_1 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a_4 & 0 & 0 \\ \hline 0 & 0 & 0 & a_5 \end{array}\right] \middle| \; a_i \in \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 54, 54.5\}, [0, 55)\right\} \subseteq V$$

is a S-special quasi vector subspace of V over the S-ring Z_{55} of finite dimension

$$M_2 = \left\{\left[\begin{array}{c|cc|c} 0 & 0 & 0 & a_1 \\ 0 & 0 & 0 & a_2 \\ \hline a_7 & 0 & 0 & 0 \\ a_3 & 0 & 0 & 0 \\ a_4 & 0 & 0 & 0 \\ \hline 0 & a_5 & a_6 & 0 \end{array}\right] \middle| \; a_1, a_2, a_3, a_4 \in [0, 55)\,, a_7, a_5, a_6 \in \{0, 0.1, 0.2, 0.3, \ldots, 0.9, 1, 1.1, \ldots, 54, 54.1, \ldots, 54.9\} \subseteq [0, 55)\right\} \subseteq V$$

be the S-special quasi vector subspace of V over the S-ring Z_{55}. M_2 is infinite dimensional S-quasi vector subspace of V over Z_{55}.

Let

$$N_1 = \left\{\left[\begin{array}{c|cc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & a_1 & a_2 & 0 \\ 0 & a_3 & a_4 & 0 \\ 0 & a_5 & a_6 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \middle| \; a_i \in \{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4,\right.$$

…, 2, …, 54, 54.2, 54.4, 54.6, 54.8} $\subseteq$ [0, 55), $1 \leq i \leq 6\} \subseteq V$

be a S-special quasi vector subspace of V of finite dimension over the S-ring Z_{55}.

Example 3.37: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \, a_i \in [0, 51); 1 \leq i \leq 9 \right\}$$

be a S-special interval pseudo linear algebra.

V is a non commutative S-special interval pseudo linear algebra under the product usual product × and is commutative S-special interval linear algebra under natural product $\times_n$.

V has both finite and infinite dimensional S-special pseudo linear subalgebras.

$$M_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \middle| \, a_i \in Z_5; 1 \leq i \leq 3 \right\} \subseteq V$$

is a S-special interval pseudo linear subalgebra of finite order over the S-ring Z_{51}.

M_1 is finite dimensional over Z_{51}.

$$M_2 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \, a_i \in [0, 51); 1 \leq i \leq 2 \right\} \subseteq V$$

is a S-special pseudo interval linear subalgebra over the S-ring $R = Z_{51}$. Clearly dimension of M_2 over R is infinite.

$$N_1 = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\right| a_i \in Z_{51}; 1 \leq i \leq 9\right\} \subseteq V \text{ is a S-special}$$

interval linear subalgebra which is non commutative over $R = Z_{51}$.

N_1 is also a finite dimensional S-special pseudo linear algebra over $R = Z_{51}$.

Consider

$$N_2 = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\right| a_i \in Z_{51}; 1 \leq i \leq 9, \times_n\right\} \subseteq V$$

is a S-special interval pseudo linear subalgebra over $R = Z_{51}$ which is commutative and N_2 is also finite dimensional over the S-ring $R = Z_{51}$.

Let

$$T_1 = \left\{\left.\begin{bmatrix} a_1 & a_2 & 0 \\ 0 & a_5 & a_6 \\ a_7 & 0 & a_9 \end{bmatrix}\right| a_i \in [0, 51); 1 \leq i \leq 9, \times_n\right\} \subseteq V$$

is a S-special pseudo linear subalgebra which is commutative and is of infinite dimension over $R = Z_{51}$.

Let

$$T_2 = \left\{\left.\begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ a_3 & 0 & a_4 \end{bmatrix}\right| a_i \in [0, 51); 1 \leq i \leq 4, \times_n\right\} \subseteq V$$

be an infinite dimensional S-special linear subalgebra of V which is commutative over $R = Z_{51}$.

Clearly under $\times_n$ all S-special subalgebras are commutative.

Example 3.38: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{bmatrix} \middle| \, a_i \in [0, 15); 1 \leq i \leq 25, \times \right\}$$

be the S-special interval linear algebra over the S-ring $Z_{15} = R$.

V is non commutative pseudo linear algebra of infinite dimension over $R = Z_{15}$.

Let

$$P_1 = \left\{ \begin{bmatrix} a_1 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & 0 & a_5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \middle| \, a_i \in [0, 15) \right\} \subseteq V$$

be the S-special quasi vector subspace of V over the S-ring Z_{15}.

Let

$$A = \begin{bmatrix} 5 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 & 5 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in P_1,$$

we find

$$A \times B = \begin{bmatrix} 5 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 7 & 5 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 10 & 1 & 0 & 0 \\ 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \notin P_1;$$

Hence P_1 is only a S-special interval quasi vector subspace of V and is not a S-special linear subalgebra of V over Z_{15}.

Let

$$P_2 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{bmatrix} \middle| \; a_i \in \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 13, 13.5, 14, 14.5\} \subseteq [0, 15); 1 \leq i \leq 15 \right\} \subseteq V$$

be the S-special quasi vector subspace of V over the S-ring Z_{15}.

P_2 is not S-special pseudo linear subalgebra of V over Z_{15}.

P_2 is only finite dimensional as a S-quasi special vector subspace of V.

$$\text{Let } A = \begin{bmatrix} 0.5 & 0.5 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 1.5 & 0.5 \\ 1.5 & 0.5 & 0.5 & 0.5 & 1 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 1 & 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix} \in P_2$$

$$A \times A = \begin{bmatrix} 0.5 & 0.5 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 1.5 & 0.5 \\ 1.5 & 0.5 & 0.5 & 0.5 & 1 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 1 & 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix} \times$$

$$\begin{bmatrix} 0.5 & 0.5 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 1.5 & 0.5 \\ 1.5 & 0.5 & 0.5 & 0.5 & 1 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 \\ 1 & 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.75 & 1 & 1.5 & 1 \\ 2 & 1.25 & 2 & 1.25 & 1.25 \\ 2.75 & 1.75 & 1.5 & 2.5 & 1.75 \\ 2.5 & 1.25 & 1.5 & 2 & 1.5 \\ 2 & 1.25 & 1.25 & 2 & 1.75 \end{bmatrix} \notin P_2.$$

Thus P_2 is not a S-special linear subalgebra only a S-special quasi vector subspace of V over Z_{15}.

Now having seen examples of finite and infinite dimensional S-special quasi vector subspaces and S-special linear subalgebra both commutative we proceed onto illustrate the notion of S-special linear transformation and S-special linear operator.

In the first place the notion of S-special linear transformation of S-special interval vector spaces and S-special interval linear algebras can be defined only if both of them are defined over the S-ring Z_n.

We will illustrate this situation by some examples.

Example 3.39: Let $V = \{(a_1, a_2, a_3, a_4) \mid a_i \in [0, 6), 1 \leq i \leq 4\}$ and

$$W = \left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}\right| a_i \in [0, 6); 1 \leq i \leq 6\right\}$$

be two S-special interval vector spaces over the S-ring Z_6.

Let $T : V \to W$ be a map such that

$$T\{(a_1, a_2, a_3, a_4)\} = \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ a_3 \\ a_4 \end{bmatrix}.$$

Clearly T is a S-special linear transformation from V to W.

We can also define $T_1 : W \to V$ by

$$T_1 \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \right\} = (a_1 + a_2, a_3, a_4, a_5 + a_6);$$

T_1 is also a S-special linear transformation from W to V.

As in case of usual linear transformations we can in case of S-special linear transformation define $\ker T_1$.

$\ker T_1 = \{x \in W \mid T_1(x) = (0)\} \neq (0, 0, 0, 0)$.

Thus ker T_1 is a non trivial subspace of W.

We can also define projections in case of S-special linear operations. Before we proceed to describe S-special linear projections and S-special linear operators we give some more examples of S-special linear transformations.

Example 3.40: Let

$$V = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \right| a_i \in [0, 26); 1 \leq i \leq 15 \right\}$$

and

$$W = \left\{ \left. \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \end{pmatrix} \right| a_i \in [0, 26); 1 \leq i \leq 12 \right\}$$

be S-special vector spaces defined over the S-ring Z_{26}.

Define T : V → W by

$$T\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix}\right) = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \end{pmatrix}.$$

T is a S-special linear transformation from V to W.

We can define T_1 : W → V by

$$T_1\left\{\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \end{pmatrix}\right\} = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ 0 & 0 & 0 \end{bmatrix};$$

T_1 is a S-special linear transformation from W to V.

Since all the S-special interval vector spaces (S-special interval pseudo linear algebras) defined over the S-ring, Z_n happens to be infinite dimensional; we have not describing results as in case of finite dimensional S-vector spaces.

Example 3.41: Let

$$V = \left\{\left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ \hline a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ \hline a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{array}\right] \middle| \; a_i \in [0, 46);\ 1 \leq i \leq 18\right\}$$

and

$$W = \left\{ \left(\begin{array}{c|cc|cccc|cc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \end{array} \right) \middle| \; a_i \in [0, 46);\; 1 \leq i \leq 18 \right\}$$

be S-special interval linear algebras defined over the S-ring Z_{46}.

Define T : V → W by

$$T\left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ \hline a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ \hline a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{array} \right] \right\}$$

$$= \left(\begin{array}{c|cc|cccc|cc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \end{array} \right);$$

T is a S-special linear transformation.

Infact T is one to one and onto.

Example 3.42: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_9 & a_{17} \\ a_2 & a_{10} & a_{18} \\ a_3 & a_{11} & a_{19} \\ a_4 & a_{12} & a_{20} \\ a_5 & a_{13} & a_{21} \\ a_6 & a_{14} & a_{22} \\ a_7 & a_{15} & a_{23} \\ a_8 & a_{16} & a_{24} \end{bmatrix} \middle| \; a_i \in [0, 44); 1 \leq i \leq 24 \right\}$$

be a S-special interval vector space over the S-ring Z_{44}.

Define T : V → V by

$$T\left(\begin{bmatrix} a_1 & a_9 & a_{17} \\ a_2 & a_{10} & a_{18} \\ a_3 & a_{11} & a_{19} \\ a_4 & a_{12} & a_{20} \\ a_5 & a_{13} & a_{21} \\ a_6 & a_{14} & a_{22} \\ a_7 & a_{15} & a_{23} \\ a_8 & a_{16} & a_{24} \end{bmatrix} \right) = \begin{bmatrix} a_1 & a_9 & a_{17} \\ 0 & 0 & 0 \\ a_3 & a_{11} & a_{19} \\ 0 & 0 & 0 \\ a_5 & a_{13} & a_{21} \\ 0 & 0 & 0 \\ a_7 & a_{15} & a_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

T is a S-special linear operator on V.

$$\ker T \neq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\text{Let } M_1 = \left\{ \left. \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ a_3 & 0 & 0 \\ a_4 & 0 & 0 \\ a_5 & 0 & 0 \\ a_6 & 0 & 0 \\ a_7 & 0 & 0 \\ a_8 & 0 & 0 \end{bmatrix} \right| a_i \in [0, 44);\ 1 \leq i \leq 8 \right\} \subseteq V;$$

the restriction of T to M_1 is given by

$$T\{M_1\} = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ a_3 & 0 & 0 \\ 0 & 0 & 0 \\ a_5 & 0 & 0 \\ 0 & 0 & 0 \\ a_7 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now we can define also the notion of projection mapping.

Let

$$W_1 = \left\{\left.\begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ a_3 & 0 & 0 \\ a_4 & 0 & 0 \\ a_5 & 0 & 0 \\ a_6 & 0 & 0 \\ a_7 & 0 & 0 \\ a_8 & 0 & 0 \end{bmatrix}\right| a_i \in [0, 44); 1 \leq i \leq 8\right\} \subseteq V$$

be the S-special vector subspace of V over the S-ring Z_{44}.

Define $T_1 : V \to V$ by

$$T_1 \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \\ a_{22} & a_{23} & a_{24} \end{bmatrix}\right\} \to \begin{bmatrix} a_1 & 0 & 0 \\ a_4 & 0 & 0 \\ a_7 & 0 & 0 \\ a_{10} & 0 & 0 \\ a_{13} & 0 & 0 \\ a_{16} & 0 & 0 \\ a_{19} & 0 & 0 \\ a_{22} & 0 & 0 \end{bmatrix}.$$

T_1 is a S-special linear operator on V; infact T_1 is a projection on V.

We see $T_1 \text{ o } T_1 = T_1$.

Consider $T_2 : V \to V$ given by

$$T_2\{\begin{bmatrix} a_1 & a_9 & a_{17} \\ a_2 & a_{10} & a_{18} \\ a_3 & a_{11} & a_{19} \\ a_4 & a_{12} & a_{20} \\ a_5 & a_{13} & a_{21} \\ a_6 & a_{14} & a_{22} \\ a_7 & a_{15} & a_{23} \\ a_8 & a_{16} & a_{24} \end{bmatrix}\} = \begin{bmatrix} 0 & a_9 & 0 \\ 0 & a_{10} & 0 \\ 0 & a_{11} & 0 \\ 0 & a_{12} & 0 \\ 0 & a_{13} & 0 \\ 0 & a_{14} & 0 \\ 0 & a_{15} & 0 \\ 0 & a_{16} & 0 \end{bmatrix}.$$

We see T_2 is a S-special linear operator on V and $T_2 \circ T_2 = T_2$.

However if

$$W_2 = \left\{\begin{bmatrix} 0 & a_1 & 0 \\ 0 & a_2 & 0 \\ 0 & a_3 & 0 \\ 0 & a_4 & 0 \\ 0 & a_5 & 0 \\ 0 & a_6 & 0 \\ 0 & a_7 & 0 \\ 0 & a_8 & 0 \end{bmatrix} \middle| \; a_i \in [0, 44); 1 \leq i \leq 8\right\} \subseteq V;$$

then T_2 can be realized as a S-special linear projection of V onto W_2.

Thus we can depending on V define S-special linear projection depending on the subspaces.

Let

$$W_3 = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}\right| a_i \in [0, 44); 1 \leq i \leq 3\right\} \subseteq V$$

be a S-special vector subspace of V over the S-ring Z_{44}.

Define $T_3 : V \to V$ by

$$T_3 \left\{\begin{bmatrix} a_1 & a_9 & a_{17} \\ a_2 & a_{10} & a_{18} \\ a_3 & a_{11} & a_{19} \\ a_4 & a_{12} & a_{20} \\ a_5 & a_{13} & a_{21} \\ a_6 & a_{14} & a_{22} \\ a_7 & a_{15} & a_{23} \\ a_8 & a_{16} & a_{24} \end{bmatrix}\right\} = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}.$$

Clearly T_3 is a S-linear operator on V. T_3 is a projection of V on W_3

$T_3 \text{ o } T_3 = T_3$.

$$\text{Let } W_4 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}\right| a_i \in [0, 44); 1 \leq i \leq 6\right\} \subseteq V$$

be a S-special vector subspace of V over the S-ring $R = Z_{44}$.

Define $T_4 : V \to V$ by

$$T_4 \left\{ \begin{bmatrix} a_1 & a_9 & a_{17} \\ a_2 & a_{10} & a_{18} \\ a_3 & a_{11} & a_{19} \\ a_4 & a_{12} & a_{20} \\ a_5 & a_{13} & a_{21} \\ a_6 & a_{14} & a_{22} \\ a_7 & a_{15} & a_{23} \\ a_8 & a_{16} & a_{24} \end{bmatrix} \right\} = \begin{bmatrix} 0 & 0 & 0 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}.$$

T_4 is a S-special linear operator on V and T_4 is a projection of V into W_4 and $T_4 \circ T_4 = T_4$.

Let

$$W_5 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \middle| \; a_i \in [0, 44); 1 \leq i \leq 6 \right\} \subseteq V$$

be a S-special vector subspace of V over the S-ring $R = Z_{44}$.

Let

$$W_6 = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_7 & a_8 & a_9 \\ 0 & 0 & 0 \end{bmatrix} \middle| \, a_i \in [0, 44); 1 \leq i \leq 9\right\} \subseteq V$$

be a S-special vector subspace of V over the S-ring R = Z_{44}.

Define $T_6 : V \rightarrow V$

$$\text{by } T_6 \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \\ a_{22} & a_{23} & a_{24} \end{bmatrix}\right\} = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_7 & a_8 & a_9 \\ 0 & 0 & 0 \end{bmatrix}.$$

T_6 is a S-linear operator on V and infact T_6 is a S-linear projection on V to W_6.

Define $T_5 : V \to V$ by

$$T_5 \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \\ a_{22} & a_{23} & a_{24} \end{bmatrix} \right\} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix};$$

T_5 is a S-linear operator on V and infact $T_5 \circ T_5 = T_5$ and T_5 is a S-linear projection on V onto W_5.

Let

$$W_7 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \, a_i \in [0, 44);\ 1 \leq i \leq 6 \right\} \subseteq V$$

is a S-special vector subspace of V over the S-ring Z_{44}.

Define $T_7 : V \to V$ by

$$T_7 \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \\ a_{22} & a_{23} & a_{24} \end{bmatrix} \right\} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

T_7 is a S linear operator on V and T_7 is a S-linear projection of V onto W_7.

$$\text{We see ker } T_i \neq \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}.$$

Thus ker T_i is a S-special subspace of V; $1 \leq i \leq 7$.

Example 3.43: Let

$$V = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & \dots & \dots & \dots & a_{10} \\ a_{11} & \dots & \dots & \dots & a_{15} \\ a_{16} & \dots & \dots & \dots & a_{20} \\ a_{21} & \dots & \dots & \dots & a_{25} \end{bmatrix} \right| a_i \in [0, 93);\ 1 \leq i \leq 25 \right\}$$

be the S-special interval vector space over the S-ring Z_{93}.

Let

$$W_1 = \left\{\begin{bmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & 0 & a_5 \end{bmatrix} \middle| \, a_i \in [0, 93);\ 1 \leq i \leq 5\right\} \subseteq V$$

be a S-special vector subspace of V over the S-ring Z_{93}.

Define $T_1 : V \to V$ by

$$T_1\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & \ldots & \ldots & \ldots & a_{10} \\ a_{11} & \ldots & \ldots & \ldots & a_{15} \\ a_{16} & \ldots & \ldots & \ldots & a_{20} \\ a_{21} & \ldots & \ldots & \ldots & a_{25} \end{bmatrix}\right\} = \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & 0 & a_5 \end{bmatrix}$$

T_1 is a S-special linear operator on V. T_1 is the projection of V onto W_1 and $T_1 \circ T_1 = T_1$.

Let

$$W_2 = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \middle| \, a_i \in [0, 93);\ 1 \leq i \leq 5\right\} \subseteq V$$

be a S-special vector subspace of V over the S-ring Z_{93}.

$T_2 : V \to V$ defined by

$$T_2\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & \dots & \dots & \dots & a_{10} \\ a_{11} & \dots & \dots & \dots & a_{15} \\ a_{16} & \dots & \dots & \dots & a_{20} \\ a_{21} & \dots & \dots & \dots & a_{25} \end{bmatrix}\} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

T_2 is a S-special linear operator on V and T_2 is a S-linear projection of V onto W_2.

Further $T_2 \text{ o } T_2 = T_2$ is a S-special idempotent linear operator.

Let

$$W_3 = \left\{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \middle| \, a_i \in [0, 93); 1 \le i \le 10\right\} \subseteq V$$

be the S-special vector subspace of V over the S-ring Z_{93}.

Define $T_3 : V \to V$ by

$$T_3\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & \dots & \dots & \dots & a_{10} \\ a_{11} & \dots & \dots & \dots & a_{15} \\ a_{16} & \dots & \dots & \dots & a_{20} \\ a_{21} & \dots & \dots & \dots & a_{25} \end{bmatrix}\} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

T_3 is a S-special linear operator on V.

T_3 is a S-special projection operator on V. $T_3 \circ T_3 = T_3$. T_3 is an idempotent operator on V.

Let

$$W_4 = \left\{\left.\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ a_1 & a_2 & \dots & a_5 \end{bmatrix}\right| a_i \in [0, 93); 1 \leq i \leq 58\right\} \subseteq V$$

be a S-special linear operator on V.

Define $T_4 : V \to V$;

$$T_4 \left\{\begin{bmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{15} \\ a_{16} & a_{17} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{25} \end{bmatrix}\right\} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 \end{bmatrix}.$$

T_4 is a S-linear projection of V onto W_4 and $T_4 \circ T_4$ so T_4 is an idempotent operator on V.

$$\text{For } T_4 \circ T_4 \left(\begin{bmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{15} \\ a_{16} & a_{17} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{25} \end{bmatrix}\right)$$

$$= T_4\,[T_4(\begin{bmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{15} \\ a_{16} & a_{17} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{25} \end{bmatrix})]$$

$$= T_4(\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 \end{bmatrix}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 \end{bmatrix}.$$

Hence $T_4 \text{ o } T_4 = T_4$.

We see $T_2 \text{ o } T_4 = (\theta)$ where θ denotes the zero transformation on V.

$$\text{Consider } T_2 \text{ o } T_4(\begin{bmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{15} \\ a_{16} & a_{17} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{25} \end{bmatrix})$$

$$= T_4\,[T_2(\begin{bmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{15} \\ a_{16} & a_{17} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{25} \end{bmatrix})]$$

$$= T_4(\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus T_2 o T_4 (X) = (0) for all X $\in$ V. Hence T_2 o T_4 is the zero S-special linear operator on V.

$$\text{Consider } T_4 \text{ o } T_2 \ (\begin{bmatrix} a_1 & a_2 & ... & a_5 \\ a_6 & a_7 & ... & a_{10} \\ a_{11} & a_{12} & ... & a_{15} \\ a_{16} & a_{17} & ... & a_{20} \\ a_{21} & a_{22} & ... & a_{25} \end{bmatrix})$$

$$= T_2 \ (T_4 \ (\begin{bmatrix} a_1 & a_2 & ... & a_5 \\ a_6 & a_7 & ... & a_{10} \\ a_{11} & a_{12} & ... & a_{15} \\ a_{16} & a_{17} & ... & a_{20} \\ a_{21} & a_{22} & ... & a_{25} \end{bmatrix}))$$

$$= T_2 \ (\begin{bmatrix} 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ a_1 & a_2 & ... & a_5 \end{bmatrix}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus T_4 o T_2 = θ is the S-special zero linear operator on V as T_4 o T_2 (X) = (0) for all X $\in$ V. Inview of all these we have the following nice theorem.

THEOREM 3.4: *If T : V → V is a S-linear operator on V and if* $W_1 + \ldots + W_n = V$ *is the direct sum then we have*

(i) $T_i : V \to V$ *such that*
$T_i (V) = W_i$, $1 \leq i \leq n$.
(ii) $T_i \circ T_j = T_j \circ T_i$ *(zero operator on V)*
$i \neq j$, $1 \leq i, j \leq n$.
(iii) $T_i \circ T_i = T_i$ *is the S-linear idempotent operator on V;* $1 \leq i \leq n$.

Proof is direct and hence left as an exercise to the reader.

Next we give examples of S-special interval polynomial rings.

Example 3.44: Let

$$V = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 26) \right\}$$

be the S-special vector space over the S-ring $R = Z_{26}$. Dimension of V over $R = Z_6$ is infinite.

Example 3.45: Let

$$V = \left\{ \sum_{i=0}^{10} a_i x^i \,\middle|\, a_i \in [0, 14) \right\}$$

be the S-special vector space over the S-ring $R = Z_{14}$. V is of infinite dimension over $R = Z_{14}$.

Clearly V is a S-special vector space over the S-ring which is not a S-special linear algebra over the S-ring Z_{14}.

For if $p(x) = a_0 + a_1x + a_2x^9$ and $q(x) = b_0 + b_1x^8 \in V$; where a_0, a_1, a_2, b_0 and $b_1 \in [0, 14)$.

$p(x) \times q(x) = a_0 + a_1x + a_2x^9 \times (b_0 + b_1x^8)$
$= a_0b_0 + a_1b_0x + a_2b_0x^9 + a_0b_1x^8 + a_1b_1x^9 + a_2b_1x^{17} \notin V$.
Hence the claim.

Example 3.46: Let

$$V = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 39) \right\}$$

be the S-special interval linear algebra over the S-ring $Z_{39} = R$. V has S-quasi subspaces and S-subalgebras.

$$W_1 = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 39) \right\} \subseteq V$$

is a S-quasi subalgebra of V of infinite dimension over $R = Z_{39}$.

$B = \{1, x, x^2, \ldots, x^n, \ldots\} \subseteq W_1$ is a basis of W_1 over $R = Z_{39}$.

However B is not a basis of V over Z_{39}.

Let

$$W_2 = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \{0, 0.5, 1, 1.5, 2, 2.5, 3, \ldots, 38, 38.5\} \subseteq [0, 39) \right\} \subseteq V;$$

W_2 is a S-special quasi vector subspace of V over the ring $R = Z_{39}$.

Clearly W_2 is not a S-special linear subalgebra as if
$p(x) = 0.5\,x$ and $q(x) = 4.5x^7 \in W_2$;
then $p(x) \times q(x) = 0.5x \times 4.5x^7 = 2.25x^8 \notin W_2$.

Thus W_2 is only a S-special quasi vector subspace of V of infinite dimension over Z_{39}.

Let $W_3 = \left\{\sum_{i=0}^{5} a_i x^i \,\middle|\, a_i \in \{0, 0.1, 0.2, 0.3, \ldots, 0.9, 1, 1.1, 1.2, \ldots, 1.9, 2, \ldots, 38.1, 38.2, \ldots, 38.9\} \subseteq [0, 39); 0 \leq i \leq 5\right\} \subseteq V$ be the S-special quasi vector subspace of V of finite dimension over Z_{39}.

It is important to observe that there is no S-special linear subalgebra of finite dimension over $R = Z_{39}$.

Example 3.47: Let

$$V = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 55)\right\}$$

be the S-special linear algebra over the S-ring $R = Z_{55}$. V is infinite dimensional linear algebra. V has no finite linear subalgebra.

V has no finite dimensional S-special linear subalgebras. V has finite dimensional S-special quasi vector subspaces as well as infinite dimensional S-vector subspaces.

We can using these S-polynomials special interval rings build S-matrix polynomials vector spaces.

This structure is exhibited by an example or two.

Example 3.48: Let

$$V = \left\{(a_1, a_2, a_3) \mid a_i \in \left\{\sum_{i=0}^{\infty} b_i x^i \,\middle|\, b_j \in [0, 22), 1 \leq i \leq 3\right\}\right.$$

be the S-special matrix polynomial interval vector space (linear algebra) over the S-ring $R = Z_{22}$.

V has both finite and infinite dimensional S-special quasi vector subspaces but only infinite dimensional S-special sublinear algebras.

Example 3.49: Let

$$V = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} \right| a_i \in \left\{ \left. \sum_{i=0}^{\infty} b_i x^i \right| b_j \in [0, 34), 1 \leq i \leq 9 \right\} \right.$$

be the S-special interval polynomial linear algebra over the S-ring Z_{34}.

V is infinite dimensional. V has finite and infinite dimensional S-special quasi vector subspaces. However all S-special linear subalgebras of infinite order.

Example 3.50: Let

$$V = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ \vdots & \vdots & \vdots & \vdots \\ a_{37} & a_{38} & a_{39} & a_{40} \end{bmatrix} \right| a_i \in \left\{ \left. \sum_{i=0}^{\infty} b_i x^i \right| b_j \in [0, 55), \right. \right.$$

$$1 \leq i \leq 40 \}$$

be a S-special interval polynomial matrix vector space (linear algebra) over the S-ring Z_{55}.

V has finite and infinite dimensional S-special.

All S-special sublinear algebras are of infinite dimension over R = Z_{55}.

It is important to note the following.

1. The concept of eigen values and eigen vectors in general cannot always be defined for S-special linear operator spaces. As the eigen values may not be always in the S-ring Z_n.
2. The concept of S-special inner product cannot be always true for the inner product may not belong to Z_n.
3. The notion of S-special linear functionals will not find its values in Z_n.

   So to over come all the draw backs we are forced to define the notion of Smarandache strong special interval vector space (linear algebra) or strong Smarandache special interval vector space (linear algebra) in the following chapter.

Other than these all the properties enjoyed by the usual vector spaces is enjoyed by S-special interval vector spaces (linear algebras). The advantage of using this new notion is that when we study vector spaces over Z_p, p a prime why not over the S-ring, Z_n.

We suggest the following problems for this chapter.

Problems:

1. Obtain some special features enjoyed by S-special interval vector spaces over the S-ring.

2. Can S-special interval vector space over the S-ring Z_n be finite dimensional?

3. Let V = ([0, 35), +) be the special interval vector space over the S-ring Z_{35}.

   (i) Find S-subspaces of V over Z_{35}.
   (ii) Can V have finite S-vector subspaces over Z_{35}?
   (iii) How many finite dimensional S-vector subspaces are there in V over Z_{35}?
   (iv) Is V finite dimensional over Z_{35}?

(v) Is it possible to write V as a direct sum of S-subspaces?

4. Let V = {[0, 46), +} be the special interval vector space over the S-ring $R = Z_{46}$.

   Study questions (i) to (v) of problem 3 for this V.

5. Let V = {[0, 69), +} be the special interval vector space over the S-ring $R = Z_{69}$.

   Study questions (i) to (v) of problem 3 for this V.

6. Distinguish between the S-special interval vector spaces and special interval vector spaces.

7. Let $V = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in [0, 58), 1 \leq i \leq 5\}$ be the S-special interval vector space over the S-ring $R = Z_{58}$.

   (i) Prove V is infinite dimensional over $R = Z_{58}$.
   (ii) Find all subspaces which are finite dimensional over R.
   (iii) Find all infinite dimensional S-special vector subspaces of V over R.
   (iv) Can V have infinite number of finite dimensional S-special vector subspaces?
   (v) Write V as a direct sum.
   (vi) Give an example of a subset S in V and its orthogonal part $S^{\perp}$.
   Prove $S^{\perp}$ is a S-special interval subspace of V.
   (vii) Show in general if W is a subspace of V, M orthogonal to W need not in general be the orthogonal complement of W in V.
   (viii) In how many ways can we write V as a direct sum of subspaces?

8. Let $V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \end{bmatrix} \right|$ $a_i \in [0, 34); 1 \leq i \leq 70\}$ be the S-special interval vector subspace of V over the S-ring $R = Z_{34}$.

   Study questions (i) to (viii) of problem 7 for this V.

9. Let $V = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \\ a_{31} & a_{32} & \dots & a_{40} \end{bmatrix} \right|$ $a_i \in [0, 96); 1 \leq i \leq 40\}$ be the S-special interval vector subspace of V over the S-ring $R = Z_{96}$.

   Study questions (i) to (viii) of problem 7 for this V.

10. Let $V = \{[0, 52), +, \times\}$ be a S-special interval linear algebra over the S-ring Z_{52}.

    (i) Show V is of infinite dimensional over Z_{52}.
    (ii) Show V has atleast one finite dimensional S-special interval linear subalgebra.
    (iii) Show V has both finite and infinite dimensional S-special quasi vector subspaces over Z_{52}.

11. Let V = $\{(a_1, a_2, a_3, a_4) \mid a_i \in [0, 42), 1 \leq i \leq 4\}$ be the S-special interval linear algebra.

    Study questions (i) to (iii) of problem 10 for this V.

12. Let V = $\left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix}\right| a_i \in [0, 122); 1 \leq i \leq 10\right\}$ be the S-special interval linear algebra.

    Study questions (i) to (iii) of problem 10 for this V.

13. Let V = $\left\{\left.\begin{bmatrix} a_1 & a_2 & \ldots & a_9 \\ a_{10} & a_{11} & \ldots & a_{18} \\ a_{19} & a_{20} & \ldots & a_{27} \end{bmatrix}\right| a_i \in [0, 77); 1 \leq i \leq 27\right\}$ be the S-special interval linear algebra over the S-ring Z_{77}.

    Study questions (i) to (iii) of problem 10 for this V.

14. Let V = $\left\{\left.\begin{bmatrix} a_1 & a_2 & \ldots & a_8 \\ a_9 & a_{10} & \ldots & a_{16} \\ \vdots & \vdots & \ldots & \vdots \\ a_{57} & a_{58} & \ldots & a_{64} \end{bmatrix}\right| a_i \in [0, 46); 1 \leq i \leq 64\right\}$ be the S-special interval vector subspace of V over the S-ring R = Z_{46}.

    Study questions (i) to (iii) of problem 10 for this V.

15. What is the algebraic structure enjoyed by the S-special interval linear operators on V; the S-special interval linear algebra?

16. Let $V = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \end{bmatrix} \middle| a_i \in [0, 39); 1 \leq i \leq 30 \right\}$ be

    the S-special interval linear algebra over the S-ring Z_{39}.

    Find the algebraic structure enjoyed by $\text{Hom}_{Z_{39}}(V, V)$.

17. Let $V = \{(a_1, a_2, \ldots, a_{10}) \mid a_i \in [0, 46), 1 \leq i \leq 10\}$ and

    $W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix} \middle| a_i \in [0, 46); 1 \leq i \leq 12 \right\}$ be two S-special

    interval vector spaces over the S-ring Z_{46}.

    Find the algebraic structure enjoyed by $\text{Hom}_{Z_{46}}(V, W)$.

18. Write W in problem 17 as direct sum of S-special interval pseudo sublinear algebras.

19. Let $V = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_7 \\ a_8 & a_9 & \ldots & a_{14} \\ \vdots & \vdots & \ldots & \vdots \\ a_{43} & a_{44} & \ldots & a_{49} \end{bmatrix} \middle| a_i \in [0, 21); 1 \leq i \leq 49 \right\}$ be

    the S-special pseudo interval linear algebra under the usual product ‘×’.

(i) Prove V is a non commutative S-special linear algebra.
(ii) Find S-special interval quasi vector subspaces of V which are finite dimensional over Z_{21}.
(iii) Find S-special interval quasi vector subspaces of V over Z_{21} of infinite dimension over Z_{21}.
(iv) Does V contain finite dimensional S-interval sublinear algebras?

20. Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \; a_i \in [0, 62), 1 \le i \le 16, +, \times_n \right\}$$

be the S-special interval linear algebra over the S-ring Z_{62}.

Study questions (i) to (iv) of problem 19 for this V.

21. Let $V_1 = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_{18} \\ a_{19} & a_{20} & \dots & a_{36} \\ a_{37} & a_{38} & \dots & a_{54} \end{bmatrix} \middle| \; a_i \in [0, 46); 1 \le i \le 54 \right\}$

be the S-special interval linear algebra over the S-ring Z_{46}.

Study questions (i) to (iv) of problem 19 for this V.

22. (i) Study $\text{Hom}_{Z_{46}}(V, V_1)$.
(ii) Is $\text{Hom}_{Z_{46}}(V, V_1)$, a S-special interval vector space over the S-ring Z_{46}?

23. Let

$$V = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 & \ldots & a_7 \\ a_8 & a_9 & a_{10} & \ldots & a_{14} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{43} & a_{44} & a_{45} & \ldots & a_{49} \end{bmatrix} \right| a_i \in [0, 111), 1 \leq i \leq 49 \right\}$$

be the S-special interval linear algebra over the S-ring $R = Z_{41}$.

(i) Study questions (i) to (iv) of problem 19 for this V.
(ii) Is $\text{Hom}_{Z_{11}}$ (V, V) a S-special interval linear algebra over the S-ring Z_{111}?

24. Let $V = \left\{ \left. \sum_{i=0}^{\infty} a_i x^i \right| a_j \in [0, 55) \right\}$ be the S-special interval linear algebra over the S-ring Z_{55}.

Study questions (i) to (iv) of problem 19 for this V.

Chapter Four

SMARANDACHE STRONG SPECIAL PSEUDO INTERVAL VECTOR SPACES

In this chapter we proceed onto define develop and describe the notion of Smarandache Strong Special interval pseudo vector spaces (linear algebra) denoted by SSS-interval vector space or SSS interval pseudo linear algebra. Throughout this chapter [0, n) is a special interval pseudo ring which is always taken as a S-ring.

DEFINITION 4.1: *Let V be a S-special interval vector space over the S-special pseudo interval ring [0, n) then we define V to be a Smarandache Special Strong pseudo interval vector space (SSS-interval vector space) over the S-special pseudo interval ring [0, n).*

We will illustrate this situation by some simple examples.

Example 4.1: Let V = {[0, 7) × [0, 7)} be a SSS-interval pseudo vector space over the S-special pseudo interval S-ring R = [0, 7).

Example 4.2: Let V = {[0, 26)} be the SSS-interval pseudo vector space over the S-special pseudo interval ring [0, 26).

Example 4.3: Let
V = $\{(a_1, a_2, a_3, a_4, a_5, a_6) \mid a_i \in [0, 21), 1 \leq i \leq 6\}$ be the SSS-interval pseudo vector space over the S-special pseudo interval ring R = [0, 21).

Example 4.4: Let

$$V = \left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix}\right| a_i \in [0, 17); 1 \leq i \leq 10\right\}$$

be the SSS-interval pseudo vector space over the S-special interval pseudo ring R = [0, 17).

Example 4.5: Let

$$V = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right| a_i \in [0, 33); 1 \leq i \leq 33\right\}$$

be the SSS-interval pseudo vector space over the S-special interval pseudo ring R = [0, 33).

Example 4.6: Let

$$V = \left\{\left.\begin{bmatrix} a_1 & a_2 & \ldots & a_{12} \\ a_{13} & a_{14} & \ldots & a_{24} \\ a_{25} & a_{26} & \ldots & a_{36} \\ a_{37} & a_{38} & \ldots & a_{48} \end{bmatrix}\right| a_i \in [0, 62); 1 \leq i \leq 48\right\}$$

be the SSS-interval pseudo vector space over the S-special interval pseudo ring R = [0, 62).

Example 4.7: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_7 \\ a_8 & a_9 & \ldots & a_{14} \\ a_{15} & a_{16} & \ldots & a_{21} \\ a_{37} & a_{38} & \ldots & a_{28} \\ a_{29} & a_{30} & \ldots & a_{35} \\ a_{36} & a_{37} & \ldots & a_{42} \\ a_{43} & a_{44} & \ldots & a_{49} \end{bmatrix} \right| a_i \in [0, 43); 1 \leq i \leq 49 \}$$

be the SSS-interval pseudo vector space over the S-special interval pseudo ring R = [0, 43).

We can define the concept of SSS- pseudo interval vector subspace and SSS-dimension of a SSS-vector space.

We will illustrate this by the following examples.

Example 4.8: Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} \right| a_i \in [0, 13); 1 \leq i \leq 8 \}$$

be the SSS-interval pseudo vector space over the S-special pseudo interval ring R = [0, 13).

$$W_1 = \left\{ \left. \begin{bmatrix} a_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right| a_1 \in [0, 13) \right\} \subseteq V$$

is a SSS-interval pseudo vector subspace of V over R = [0, 13).

$$W_2 = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right| a_i \in [0, 13), 1 \leq i \leq 3 \right\} \subseteq V$$

is a SSS-interval pseudo vector subspace of V over R = [0, 13).

$$W_3 = \left\{ \left. \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \right| a_i \in [0, 13), 1 \leq i \leq 3 \right\} \subseteq V$$

is a SSS-interval vector pseudo subspace of V over the S-special interval pseudo ring R = [0, 13).

$$W_4 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ a_2 \\ 0 \\ a_3 \\ 0 \\ a_4 \\ 0 \end{bmatrix} \middle| \; a_i \in [0, 13), 1 \leq i \leq 4 \right\} \subseteq V$$

is again a SSS-interval pseudo vector subspace of V over R = [0, 13).

Example 4.9: Let
$V = \{(a_1, a_2, a_3, \ldots, a_{15}) \mid a_i \in [0, 22), 1 \leq i \leq 15\}$ be the SSS-interval pseudo vector space over the S-special pseudo interval ring R = [0, 22).

$W_1 = \{(a_1, a_2, a_3, 0, \ldots, 0) \mid a_i \in [0, 22), 1 \leq i \leq 3\} \subseteq V$ is a SSS-interval pseudo vector subspace of V over R = [0, 22).

$W_2 = \{(0, 0, 0, 0, 0, a_1, a_2, a_3, a_4, a_5, 0, 0, 0, 0, 0) \mid a_i \in [0, 22), 1 \leq i \leq 5\} \subseteq V$ is a SSS-interval pseudo vector subspace of V over the ring R = [0, 22).

$W_3 = \{(0, 0, \ldots, 0, a_1, a_2) \mid a_1\ a_2 \in [0, 22)\} \subseteq V$ is a SSS-interval vector pseudo subspace of V over the ring R = [0, 22).

Example 4.10: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \middle| \; a_i \in [0, 6), 1 \leq i \leq 30 \right\}$$

be the SSS-interval pseudo vector space over the S-special pseudo interval ring R = [0, 6).

$$W_1 = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ a_4 & a_5 & a_6 \end{bmatrix}\right| a_i \in [0, 6), 1 \leq i \leq 6\right\} \subseteq V$$

is the SSS-interval pseudo vector space over the S-special pseudo interval ring R = [0, 6).

$$W_2 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}\right| a_i \in [0, 26), 1 \leq i \leq 9\right\} \subseteq V$$

is the SSS-interval pseudo vector space over the S-special pseudo interval ring R = [0, 6).

Example 4.11: Let

$$V = \left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix}\right| a_i \in [0, 43), 1 \leq i \leq 9\right\}$$

be the SSS-linear pseudo algebra over the S- pseudo ring R = [0, 43).

V is finite dimensional over R. V has several SSS-linear pseudo subalgebras of V over the S-ring R = [0, 43).

$$W_1 = \left\{\left.\begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right| a_1 \in [0, 43)\right\} \subseteq V$$

$$W_2 = \left\{\left.\begin{bmatrix} 0 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right| a_2 \in [0, 43)\right\} \subseteq V \text{ and so on.}$$

$$W_9 = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ a_9 \end{bmatrix}\right| a_9 \in [0, 43)\right\} \subseteq V$$

are the nine SSS-interval linear pseudo subalgebras of V each of dimension one over R = [0, 43).

$$M_1 = \left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right| a_1, a_2 \in [0, 43)\right\} \subseteq V,$$

$$M_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ a_1 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \; a_1, a_2 \in [0, 43) \right\} \subseteq V,$$

$$M_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_1, a_2 \in [0, 43) \right\} \subseteq V \text{ and}$$

$$M_4 = \left\{ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \middle| \; a_i \in [0, 43),\ 1 \leq i \leq 3 \right\} \subseteq V$$

are the four SSS-linear pseudo subalgebras of V over the S-ring [0, 43).

M_1, M_2 and M_3 are two dimension SSS-linear pseudo subalgebras of V over R = [0, 43).

M_4 is of dimension three over R = [0, 43).

Let

$$P_1 = \left\{ \left[\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \\ 0 \\ \vdots \\ 0 \end{array}\right] \middle| \; a_i \in [0, 43), 1 \leq i \leq 4 \right\} \subseteq V,$$

be the SSS linear pseudo subalgebra of dimension four over the S-ring R = [0, 43).

$$P_2 = \left\{ \left[\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ 0 \\ \vdots \\ 0 \end{array}\right] \middle| \; a_i \in [0, 43), 1 \leq i \leq 5 \right\} \subseteq V$$

is a SSS-linear pseudo subalgebra of dimension five over [0, 43).

$$P_3 = \left\{\left.\begin{bmatrix} a_1 \\ 0 \\ a_2 \\ a_3 \\ a_4 \\ 0 \\ a_5 \\ a_6 \\ 0 \end{bmatrix}\right| a_i \in [0, 43), 1 \leq i \leq 6\right\} \subseteq V$$

is a SSS-linear pseudo subalgebra of dimension six over the S-ring [0, 43).

$$P_4 = \left\{\left.\begin{bmatrix} 0 \\ a_1 \\ a_2 \\ \vdots \\ a_7 \\ 0 \end{bmatrix}\right| a_i \in [0, 43), 1 \leq i \leq 7\right\} \subseteq V$$

is a SSS-linear pseudo subalgebra of dimension seven over the S-special pseudo ring.

$$P_5 = \left\{\left.\begin{bmatrix} 0 \\ a_1 \\ \vdots \\ 0 \\ a_8 \end{bmatrix}\right| a_i \in [0, 43), 1 \leq i \leq 8\right\} \subseteq V$$

is a SSS- pseudo linear algebra of dimension eight over the S-pseudo ring.

The SSS- pseudo linear algebra which is of dimension nine over the S-ring R = [0, 43) has SSS-linear pseudo subalgebras of all dimensions between one and eight.

Example 4.12: Let

$$V = \left\{ \left[\begin{array}{ccc} a_1 & a_5 & a_9 \\ a_2 & a_6 & a_{10} \\ a_3 & a_7 & a_{11} \\ a_4 & a_8 & a_{12} \end{array}\right] \middle| \, a_i \in [0, 29), 1 \leq i \leq 12, +, \times_n \right\}$$

be the SSS-linear pseudo algebra over the S-ring R = [0, 29).

Dimension of V over R is 12. V has SSS-linear pseudo subalgebra of various dimensions.

V is a usual vector space (linear algebra) over the field $Z_{29} \subseteq [0, 29)$.

Several interesting properties can be derived. However this V has no SSS quasi pseudo vector subspaces.

$V = W_1 + W_2 + W_3$ where

$$W_1 = \left\{ \left[\begin{array}{ccc} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ a_3 & 0 & 0 \\ a_4 & 0 & 0 \end{array}\right] \middle| \, a_i \in [0, 29), 1 \leq i \leq 4 \right\} \subseteq V,$$

$$W_2 = \left\{ \left[\begin{array}{ccc} 0 & a_1 & 0 \\ 0 & a_2 & 0 \\ 0 & a_3 & 0 \\ 0 & a_4 & 0 \end{array}\right] \middle| \, a_i \in [0, 29), 1 \leq i \leq 4 \right\} \subseteq V \text{ and}$$

$$W_3 = \left\{\left.\begin{bmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & a_4 \end{bmatrix}\right| a_i \in [0, 29), 1 \leq i \leq 4\right\} \subseteq V$$

are SSS-linear pseudo subalgebra and $V = W_1 + W_2 + W_3$.

Let

$$M_1 = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right| a_i \in [0, 29), 1 \leq i \leq 4\right\} \subseteq V,$$

$$M_2 = \left\{\left.\begin{bmatrix} 0 & 0 & a_1 \\ a_3 & a_4 & a_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right| a_i \in [0, 29), 1 \leq i \leq 4\right\} \subseteq V,$$

$$M_3 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ 0 & a_1 & a_2 \\ a_3 & a_4 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right| a_i \in [0, 29), 1 \leq i \leq 4\right\} \subseteq V,$$

$$M_4 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ a_4 & 0 & 0 \end{bmatrix}\right| a_i \in [0, 29), 1 \leq i \leq 4\right\} \subseteq V \text{ and}$$

$$M_5 = \left\{ \left. \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_1 \\ a_2 & a_3 & a_4 \end{bmatrix} \right| a_i \in [0, 29), 1 \leq i \leq 4 \right\} \subseteq V$$

are SSS-interval pseudo linear subalgebras of V over R = [0, 29).

We see

$$M_i \cap M_j \neq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ if } i \neq j,\ 1 \leq i, j \leq 5.$$

Thus $V \subseteq M_1 + M_2 + M_3 + M_4 + M_5$. Also M_i is not orthogonal to any one of the M_j's; $i \neq j$. We see W_1 is the orthogonal to W_2 but W_2 is not the orthogonal complement of W_1.

$$\text{Let } P_1 = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right| a_i \in [0, 29), 1 \leq i \leq 5 \right\} \subseteq V$$

be the SSS-interval linear pseudo subalgebra of V over the S-special interval ring R = [0, 29).

$$P_2 = \left\{ \left. \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a_1 \\ a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 \end{bmatrix} \right| a_i \in [0, 29), 1 \leq i \leq 7 \right\} \subseteq V$$

is the SSS-interval pseudo linear subalgebra of V over the ring R = [0, 29).

We see P_1 is the orthogonal complement of P_2 and vice versa.

Further $P_1 + P_2 = V$ and

$$P_1 \cap P_2 = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\} \text{ so } P_1 \text{ and } P_2 \text{ gives the direct sum of V.}$$

Now having seen examples of SSS-interval pseudo linear algebras direct sum and SSS-interval pseudo sublinear algebras; we now proceed onto define only special properties and we are not interested in studying other details.

We are more interested in other properties which we are not in a position to impose in case of S-special pseudo interval linear algebras.

Let V be a SSS-linear pseudo algebra over the S-ring R = [0, n).

On V we define the notion of pseudo inner product for if x, y ∈ V $\langle x, y\rangle$ is the pseudo inner product $\langle x, y\rangle : V \to R$; we see $\langle x, y\rangle = 0$ even if $x \neq 0$ and $y \neq 0$. x, y ∈ V all other properties remain the same.

This includes $\langle x, x\rangle = 0$ even if $x \neq 0$.

Thus by defining SSS-interval pseudo linear algebras V; we can define the pseudo inner product.

We can also define on SSS-interval pseudo linear algebra the notion of SSS-eigen values, SSS-eigen vectors and SSS-characteristic polynomials.

Further we can define SSS-linear functionals using SSS-interval linear algebra V.

All these concepts will be described only by examples.

Example 4.13: Let
$V = \{(a_1, a_2, a_3, a_4)$ where $a_i \in [0, 15), 1 \leq i \leq 4\}$ be a SSS-pseudo linear algebra over the S-special interval pseudo ring $R = [0, 15)$.

We define SSS-linear functional on V as follows:

$f_{sss} : V \rightarrow [0, 15)$ so that f_{sss} can also be realized as a SSS-linear transformation of V to [0, 15) as [0, 15) can be realized as a SSS- pseudo vector space of dimension one over [0, 15).

$f_{sss} : V \rightarrow [0, 15)$ is a SSS-linear functional; if $x = (0.112, 3.001, 4.0007, 8) \in V$ define $f_{sss}(x) = a_1 \times 0.112 + a_2 \times 3.001 + a_3 \times 4.0007 + a_4 \times 8$ where $a_i \in [0, 15), 1 \leq i \leq 4$.

We see if V* = {Collection of all SSS-linear functionals on V} then V* is also a SSS-interval vector space over [0, 15).

All this study can be derived with simple and appropriate modifications. It is also left as an exercise to the reader to prove dim V* = dim V.

We define SSS-annihilator of a subset S of a SSS-vector space V is the set S^o of SSS-linear functionals f_{sss} on V such that $f_{sss}(\alpha) = 0$ for every $\alpha \in S$.

The SSS-subset S^o of V* is a SSS- pseudo vector subspace of V.

The following theorems can be proved by the interested reader.

THEOREM 4.1: *Let V be a finite dimensional SSS-pseudo interval vector space over the S-special pseudo interval ring R = [0, n). W be a SSS-subspace of V.*

Then dim W + dim W^o *= dim V.*

THEOREM 4.2: *Let V be a finite dimensional SSS-vector pseudo space over the S-special interval pseudo S-ring [0, n).*

For each vector α *in V define* $L_\alpha (f_{sss}) = f_{sss} (\alpha)$ *;* f_{sss} *in V*. Then the mapping* $\alpha \rightarrow L_\alpha$ *is an isomorphism of V into V**.*

THEOREM 4.3: *Let* g_{sss}, f^1_{sss}, f^2_{sss}, …, f^r_{sss} *be SSS-linear functionals on a SSS- pseudo vector space V with respect to the SSS-null space* N_{sss}, N^1_{sss}, N^2_{sss}, …, N^r_{sss} *respectively. Then* g_{sss} *is a linear combination of* f^1_{sss}, f^2_{sss}, …, f^r_{sss} *if and only if* N_{sss} *contains the intersection* $N^1_{sss} \cap N^2_{sss} \cap \ldots \cap N^r_{sss}$.

Now we can as in case of usual vector spaces define SSS-eigen values etc. Let A = (a_{ij}) n × n be a n × n matrix $a_{ij} \in [0, m)$, $1 \leq i, j \leq n$.

The SSS pseudo characteristic value of A in [0, m) is a scalar c in [0, m) such that the matrix (A – CI) is non invertible.

C is the SSS pseudo characteristic value of A if and only if det (A-CI) = 0 or equivalently det (CI – A) = 0, we form the matrix (xI – A) with polynomial entries and consider polynomial f(x) = det (xI – A).

Clearly the SSS-characteristic values of A in [0, m) are just the scalars C in [0, m) such that f(C) = 0.

For this reason f is called the SSS- pseudo characteristic polynomial of A.

Here also f is monic and $f(x) \in \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, m) \right\}$.

All properties associated with characteristic polynomials are true in case of these SSS-polynomials.

Example 4.14: Let

$$V = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_7 \end{bmatrix} \right| a_i \in [0, 6), 1 \leq i \leq 6 \right\}$$

be the SSS-interval pseudo vector space over the S-special pseudo interval ring R = [0, 6) .

Define

$$f_{sss} : V \to [0, 6) \text{ by } f_{sss} \left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_7 \end{bmatrix} \right) = a_1 + a_2 + \ldots + a_7 \ (\text{mod } 6).$$

f_{sss} is a linear functional on V.

$$\text{For instance if } x = \begin{bmatrix} 3.002 \\ 4.701 \\ 3.0175 \\ 2.0016 \\ 0.90121 \\ 5.03215 \\ 1.3141 \end{bmatrix} \in V.$$

$$f_{sss}(x) = f_{sss}\left(\begin{bmatrix} 3.002 \\ 4.701 \\ 3.0175 \\ 2.0016 \\ 0.90121 \\ 5.03215 \\ 1.3141 \end{bmatrix}\right) =$$

$3.002 + 4.701 + 3.0175 + 2.0016 + 0.90121 + 5.03215 + 1.3141 = 1.96956 \in [0, 6)$.

This is the way f_{sss} is a SSS-linear functional on V.

Interested reader can form any number of such SSS-linear functionals on SSS- pseudo vector spaces over [0, n).

Example 4.15: Let

$$V = \left\{\begin{bmatrix} a_1 & \ldots & a_6 \\ a_7 & \ldots & a_{12} \\ a_{13} & \ldots & a_{18} \\ a_{19} & \ldots & a_{24} \\ a_{25} & \ldots & a_{30} \\ a_{30} & \ldots & a_{36} \end{bmatrix} \middle| \; a_i \in [0, 15), 1 \leq i \leq 36, +, \times\right\}$$

be the SSS- pseudo interval linear algebra over the S-special interval ring R = [0, 15).

Define $f_{sss} : V \rightarrow [0, 15)$ by
$f_{sss}(A) = a_{11} + a_{22} + \ldots + a_{66} \pmod{15}$ where $A = (a_{ij}) \in V$ that is $f_{sss}(A) = \text{trace } A$.

f_{sss} is SSS-linear functional on V.

Suppose

$$A = \begin{bmatrix} 0.132 & 0 & 9 & 6 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0.92 & 0 & 1.31 & 0 & 0 & 0 \\ 7.52 & 6.3 & 0 & 4.31 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3.101 & 0 \\ 0 & 0 & 7.31 & 0 & 0 & 7.1 \end{bmatrix} \in V.$$

f_{sss} (A) = trace A

$$\begin{aligned} &= 0.132 + 0 + 1.41 + 4.31 + 3.101 + 7.1 \\ &= 15.953 \pmod{15} \\ &= 0.953 \in [0, 15). \end{aligned}$$

f_{sss} is a SSS-linear functional on V.

Now having seen examples of SSS-linear functionals we now proceed onto define more properties of SSS-linear algebras.

Example 4.16: Let

$$V = \left\{ \left[\begin{matrix} a_1 & \ldots & a_6 \\ a_7 & \ldots & a_{12} \\ a_{13} & \ldots & a_{18} \\ a_{19} & \ldots & a_{24} \\ a_{25} & \ldots & a_{30} \\ a_{30} & \ldots & a_{36} \end{matrix} \right] \right| a_i \in [0, 19), 1 \leq i \leq 40\}$$

be a SSS-vector space over the S-special pseudo interval ring R = [0, 19).

Define $f_{sss} : V \to [0, 19)$ as

f_{sss} (A) = sum of column two + sum of column four (mod 19).

$= (a_1 + a_6 + a_{10} + a_{14} + a_{18} + a_{22} + a_{26} + a_{30} + a_{34} + a_{38}) + (a_4 + a_8 + a_{12} + a_{16} + a_{20} + a_{24} + a_{28} + a_{32} + a_{36} + a_{40})$ (mod 19).

f_{sss} is a SSS-linear functional on V.

Example 4.17: Let

$$V = \left\{\left.\begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \end{bmatrix}\right| a_i \in [0, 23), 1 \leq i \leq 30\right\}$$

be the SSS-linear algebra over the S-special pseudo interval ring R = [0, 23).

Define $f_{sss} : V \to [0, 23)$ as f_{sss} (A) = sum of the 3rd row

$= a_{11} + a_{12} + a_{13} + \ldots + a_{20}$.

f_{sss} is a SSS-linear functional on V.

Example 4.18: Let

$$A = \begin{bmatrix} 0.001 & 0 & 2 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.03 \end{bmatrix} \text{ with elements from } [0, 3).$$

We find the SSS- pseudo characteristic polynomial associated with A.

$$|Ix - A| = \left|\begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix} - \begin{bmatrix} 0.001 & 0 & 2 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.03 \end{bmatrix}\right|$$

$$= \left| \begin{bmatrix} x-0.001 & 0 & 2 \\ 0 & x-0.04 & 0 \\ 0 & 0 & x-0.03 \end{bmatrix} \right|$$

$$= (x + 2.999)\ (x + 2.96)\ (x + 2.97)$$

$$= 0.$$

$x = 0.001, 0.04$ and 0.03.

Thus the SSS- pseudo eigen values of A are 0.001, 0.04 and 0.03. Now we can find SSS- pseudo eigen values for any square matrix with entries from [0, n). If the values are real we get these SSS- pseudo eigen values.

Example 4.19: Let $V = \{(a_1, a_2, a_3) \mid a_i \in [0, 5), 1 \leq i \leq 3\}$ be SSS-interval pseudo vector space over the S-special pseudo interval ring R = [0, 5).

We define

$\langle x, y\rangle_{sss} : V \times V \to [0, 5)$ as if $x = (0.0221, 0.31, 0.7)$ and $y = (0.01, 0.04, 0.071) \in V$ then $\langle x, y\rangle_{sss} = (0.0221, 0.31, 0.7) \times (0.01, 0.04, 0.071)$

$$\begin{aligned} &= (0.0221 \times 0.01 + 0.31 \times 0.04 + 0.7 \times 0.071) \\ &= 0.000221 + 0.0124 + 0.0497 \\ &= 0.062321 \in [0, 5). \end{aligned}$$

This is the way the inner product is defined.

Let $x = (2, 1, 1)$ and $y = (1, 3, 0) \in V$.

$$\begin{aligned} \langle x, y\rangle_{sss} &= \langle(2, 1, 1), (1, 3, 0)\rangle \\ &= 2 + 3 + 0 \pmod 5 \\ &= 5. \end{aligned}$$

Thus x is orthogonal to y.

Let V be SSS-vector space.

If on V we have an inner product $\langle x, y\rangle_{sss}$ defined then we call V to be SSS-inner product space over R = [0, n) ($n < \infty$).

Now if V is a SSS- pseudo inner product space over the S-special interval pseudo ring R = [0, n), we say for any $W \subseteq V$, W the SSS- pseudo vector subspace of V the orthogonal complement W to be $W^{\perp} = \{x \in V \mid \langle x, y\rangle = 0 \text{ for all } y \in W\}$.

We will illustrate this situation by some examples.

Example 4.20: Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| \, a_i \in [0, 21), 1 \leq i \leq 6 \right\}$$

be a SSS-vector space over the S-special pseudo interval ring R = [0, 21).

Let V be an inner SSS- pseudo product space where $\langle x, y\rangle_{sss}$

is defined by $W^{\perp} = \left\{ \begin{bmatrix} 0 \\ a_1 \\ 0 \\ a_2 \\ a_3 \\ 0 \end{bmatrix} \middle| \, a_i \in [0, 21), 1 \leq i \leq 3 \right\} \subseteq V$ is such

that for every $x \in W$ and for $y \in W^{\perp}$ $\langle x, y\rangle_{sss} = 0 \in [0, 21)$. $W^{\perp} = \{x \in V \mid \langle x, y\rangle_{sss} = 0 \text{ for all } y \in W\}$.

Thus we have SSS-orthogonal vectors.

$$W \cap W^{\perp} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus if $x = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}$ and $y = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix} \in V$ then

$$\langle x, y\rangle_{sss} = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 + a_5b_5 + a_6b_6 \pmod{21}.$$

Let $W = \left\{ \begin{bmatrix} a_1 \\ 0 \\ a_2 \\ 0 \\ 0 \\ a_3 \end{bmatrix} \middle| \, a_i \in [0, 21), 1 \leq i \leq 3 \right\} \subseteq V$

be a SSS- pseudo vector subspace of V over the pseudo ring R = [0, 21).

$$W^{\perp} = \left\{ \begin{bmatrix} 0 \\ a_1 \\ 0 \\ a_2 \\ a_3 \\ 0 \end{bmatrix} \middle| \; a_i \in [0, 21), 1 \leq i \leq 3 \right\} \subseteq V$$

is the orthogonal with W with $W \cap W^{\perp} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ answer is yes?

Now suppose S is only a subset in V, V a SSS-inner product space.

What will be $S^{\perp}$.

$$\text{Let } S = \left\{ \begin{bmatrix} 0.7 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.5 \\ 0 \end{bmatrix} \right\} \subseteq V$$

be a subset of V.

To find $S^{\perp} = \{x \in V \mid \langle x, y \rangle_{sss} = 0 \text{ for all } y \in S\}$

$$S^{\perp} = \left\{ \left. \begin{bmatrix} 0 \\ a_1 \\ a_2 \\ a_3 \\ 0 \\ a_4 \end{bmatrix} \right| a_i \in V, 1 \leq i \leq 4 \right\} \subseteq V$$

is the orthogonal to S of SSS- pseudo subspace of V.

$$\text{Clearly } S \cap S^{\perp} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ but however } S + S^{\perp} \neq V.$$

Thus the orthogonal complement of a subset is also a SSS- pseudo subspace of V.

Example 2.21: Let

$$V = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 \\ \vdots & \vdots & \vdots \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \right| a_i \in [0, 41), 1 \leq i \leq 15 \right\} \subseteq V$$

be the SSS- pseudo vector space of over the S-special pseudo interval ring R = [0, 41).

$$\text{Define } \langle x, y\rangle_{sss} : V \to [0, 41) \text{ by } \langle A, B\rangle_{sss} = \sum_{i=1}^{15} a_i b_i \pmod{41}$$

$$\text{where } A = \begin{bmatrix} a_1 & a_2 & a_3 \\ \vdots & \vdots & \vdots \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 & b_2 & b_3 \\ \vdots & \vdots & \vdots \\ b_{13} & b_{14} & b_{15} \end{bmatrix} \in V.$$

$\langle A, B\rangle_{sss}$ is a SSS-inner product on the SSS- pseudo vector space V over special pseudo R = [0, 41).

$$\text{Let } A = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \in V$$

we see there exists infinitely many B $\in$ V such that $\langle A, B\rangle_{sss} = 0$.

So even for a single element S = {A} we see

$$S^{\perp} = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ \vdots & \vdots & \vdots \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \, a_i \in [0, 41), 1 \leq i \leq 12 \right\} \subseteq V.$$

$S^{\perp}$ is a SSS- pseudo vector subspace of V. $S^{\perp} + S \neq V$.

$$\text{But } S \cap S^{\perp} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}.$$

Interested reader can get the analogue of the Gram Schmidt process in case of SSS- pseudo vector space with some direct and appropriate modifications.

THEOREM 4.4: *Let W be a SSS-finite pseudo dimensional subspace of a SSS-inner product space V over the S-special interval pseudo ring R = [0, n). Let E_{SSS} be a SSS orthogonal projection of V on W.*

Then E_{SSS} is an idempotent SSS-linear transformation of V onto W, $W^{\perp}$ is the SSS null space of E and $V = W \oplus W^{\perp}$.

Proof is similar to as that of usual spaces hence left as an exercise to the reader.

Next we proceed onto define the notion of SSS- pseudo polynomial vector space over S- pseudo special interval ring [0, n).

DEFINITION 4.2: *Let*

$$V = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, n),\ n < \infty \right\}$$

be the SSS-polynomial pseudo vector space defined over the S-special pseudo interval ring R = [0, n). V is an infinite dimensional SSS vector space over R. V is also a SSS- pseudo linear algebra over R.

We will first illustrate this situation by examples.

Examples 4.22: Let

$$V = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 43) \right\}$$

be a SSS-polynomial pseudo vector space over the S-special interval S- pseudo interval ring R = [0, 43).

Let V be a SSS- pseudo linear algebra under the usual product of polynomials over the S- pseudo special interval ring R = [0, 43).

Let W = $\left\{\sum_{i=0}^{11} a_i x^i \,\middle|\, a_i \in [0, 43), 0 \leq i \leq 11\right\} \subseteq V$ is a SSS quasi pseudo vector subspace of V and W is not a SSS- pseudo linear subalgebra of V as product of two polynomials is not defined in W.

Only in case of SSS polynomial linear pseudo algebras alone we are in a position to define SSS- pseudo quasi vector subspaces of V.

Almost all properties associated with usual vector spaces can be extended in case of SSS-vector spaces with some appropriate modifications.

We suggest some problems for the reader.

Some of the problems are difficult at research level and some of them are simple and some are little hard and consume more time.

Problems

1. Obtain some special features enjoyed by SSS- pseudo interval vector spaces over the S- pseudo ring [0, n) ($n < \infty$).

2. Spell out some of the advantages of using SSS- pseudo interval spaces in the place of S-interval pseudo special vector spaces.

3. Is it possible to define S-linear functionals using S-special interval pseudo linear vector spaces?

4. Let V = $\{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in [0, 46), 1 \leq i \leq 5\}$ be the SSS- pseudo linear algebra over the S- pseudo special interval ring R = [0, 46).

   (i) Find dimension of V over R.

(ii) Does we have infinite number of basis for V?
(iii) Find all SSS- pseudo subspaces of dimension two over R.
(iv) Find W a SSS- pseudo subspace of V so that

i. $W^{\perp}$ is its orthogonal complement.
ii. $W_1^{\perp}$ is just orthogonal with W and is not the orthogonal complement of W.

5. Let W = $\left\{\left[\begin{array}{cccc} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{15} \\ a_{16} & a_{17} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{25} \end{array}\right] \right|$ $a_i \in [0, 7), 1 \leq i \leq 25\}$ be the SSS- pseudo special interval vector space (linear algebra under × or $\times_n$) over the ring R = [0, 7).

(i) Study questions (i) to (iv) of problem 4 for this V.
(ii) Prove under × V is a non commutative SSS- pseudo linear algebra.

6. Let M = $\{(a_1, a_2, \dots, a_{11}) \mid a_i \in [0, 18), 1 \leq I \leq 11\}$ be a SSS- pseudo linear algebra over the S-special interval ring R = [0, 18).

7. Let P = $\left\{\left[\begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_{15} \end{array}\right] \right|$ $a_i \in [0, 43); 1 \leq i \leq 15\}$ be a SSS- pseudo linear algebra over the S-special interval ring R = [0, 43).

(i) Study questions (i) to (iv) of problem 4 for this S.

(ii) Find the algebraic structure enjoyed by $\mathrm{Hom}_R(V, V)$.

8. Let $V = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_9 \\ a_{10} & a_{11} & \ldots & a_{18} \\ a_{19} & a_{20} & \ldots & a_{27} \\ \vdots & \vdots & \vdots & \vdots \\ a_{73} & a_{74} & \ldots & a_{81} \end{bmatrix} \middle| \; a_i \in [0, 46); 1 \leq i \leq 81 \right\}$

be a SSS-non commutative pseudo linear algebra over the S-special pseudo interval ring R = [0, 46).

(i) Study questions (i) to (iv) of problem 4 for this S.

(ii) What is the distinct feature enjoyed by V as V is a SSS-non commutative linear algebra?

9. Let $V = \{(a_1, a_2, \ldots, a_{10}) \mid a_i \in [0, 23)\}$ be a SSS- pseudo linear algebra over the S-special pseudo interval ring R = [0, 23).

(i) How many distinct inner products be defined on V?

(ii) What is the dimension of V as a SSS- pseudo vector space over R?

(iii) What is the dimension of V as a SSS- pseudo linear algebra over R?

(iv) In how many ways can V be written as a direct sum?

10. Let V = $\left\{\left.\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \\ a_{13} & a_{14} \\ a_{15} & a_{16} \\ a_{17} & a_{18} \end{bmatrix} \right| a_i \in [0, 48); 1 \leq i \leq 18\right\}$ be SSS-pseudo linear algebra over the S-special pseudo interval ring R = [0, 48) under the natural product $\times_n$ of matrices.

    Study questions (i) to (iv) of problem 9 for this V.

11. Let V = $\left\{\left.\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{pmatrix} \right| a_i \in [0, 35); 1 \leq i \leq 10\right\}$ be a SSS- pseudo linear algebra over the S- pseudo ring R = [0, 35).

    Study questions (i) to (iv) of problem 9 for this V.

12. Describe some special features enjoyed by SSS- pseudo linear functionals on a SSS- pseudo linear algebra over R = [0, n).

13. Obtain Bassel's inequality for SSS-inner product spaces.

14. Give any other special feature associated with SSS-inner product space.

15. Give some special properties enjoyed by SSS-linear functionals on V; V a SSS- pseudo vector space over a ring [0, n) $n < \infty$.

16. Let $V = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 22) \right\}$ be a SSS- pseudo linear algebra over the S-special pseudo interval ring R=[0, 22).

    (i) What is dimension of V, a SSS- pseudo vector space over R = [0, 22)?
    (ii) What is the dimension of V as a SSS- pseudo linear algebra over R = [0, 22)?
    (iii) Show V can have SSS-quasi pseudo vector subspaces over R = [0, 22).
    (iv) Can a inner product $\langle \; \rangle_{SSS}$ be defined on V?
    (v) Can a SSS-linear functional be defined on V?

17. Let $V = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 29) \right\}$ be a SSS- pseudo linear algebra over the S-special pseudo interval ring R = [0, 29).

    Study questions (i) to (v) of problem 16 for this V.

18. Let $V = \left\{ \sum_{i=0}^{20} a_i x^i \,\middle|\, a_i \in [0, 13),\ 0 \leq i \leq 20 \right\}$ be a SSS- pseudo linear algebra over the S-special interval pseudo ring R = [0, 13).

    (i) Prove W is only a SSS- pseudo vector space and is not a SSS- pseudo linear algebra over R = [0, 13).
    (ii) Find a basis of W over R.
    (iii) Is W finite dimensional?
    (iv) Can W have SSS-vector subspaces?
    (v) Can a SSS-inner product be defined on W?

(vi) Find
W* = {Collection of all SSS-linear functions on W}.

(vii) Can SSS-linear operators be defined on W?

19. Let $M = \left\{ \sum_{i=0}^{12} a_i x^i \,\middle|\, a_i \in [0, 11), 0 \leq i \leq 12 \right\}$ be a SSS-linear pseudo algebra over the S-special interval pseudo ring R = [0, 11).

Study questions (i) to (vii) of problem 18 for this M.

20. Let $N = \left\{ \sum_{i=0}^{7} a_i x^i \,\middle|\, a_i \in [0, 5), 0 \leq i \leq 7 \right\}$ be a SSS-linear pseudo algebra over the S-special pseudo interval ring R = [0, 5).

Study questions (i) to (vii) of problem 18 for this N.

21. Let V = {($a_1, a_2, \ldots, a_{12}$) | $a_i \in [0, 11)$,. $1 \leq i \leq 12$} and

$$W = \left\{ \begin{bmatrix} a_1 & a_2 \\ \vdots & \vdots \\ a_{11} & a_{12} \end{bmatrix} \,\middle|\, a_i \in [0, 11); 1 \leq i \leq 12 \right\}$$ be two

SSS-vector space over the S-special pseudo interval ring R = [0, 11).

(i) Find Hom_R (V, W).

(ii) What is the algebraic structure enjoyed by $\text{Hom}_R(V_1, W)$?

(iii) Is Hom_R (V, W) a SSS- pseudo vector space over R?

(iv) Find H_R (V, V) and $\text{Hom}_R(W, W)$.

(v) Is $H_R (V, V) \cong \text{Hom}_R(W, W)$?

(vi) Is $\text{Hom}_R(V, V)$ and $\text{Hom}_R(W, W)$ SSS- pseudo linear algebra over R of same finite dimension.

22. Let $V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & \cdots & \cdots & \cdots & a_{12} \\ a_{13} & a_{14} & \cdots & \cdots & \cdots & a_{18} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{31} & a_{32} & \cdots & \cdots & \cdots & a_{36} \end{bmatrix} \right|$ $a_i \in [0, 19)$;

$1 \leq i \leq 36\}$ be the SSS- pseudo linear algebra under × and $W = \left\{ \begin{pmatrix} a_1 & \cdots & a_{18} \\ a_{19} & \cdots & a_{36} \end{pmatrix} \right|$ $a_i \in [0, 19)$; $1 \leq i \leq 36\}$

be SSS- pseudo linear algebra.

Study questions (i) to (vi) of problem 21 for this V and W.

23. Let $V = \left\{ \left(\begin{array}{cc|ccc|c} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \end{array} \right) \right|$ $a_i \in [0, 14)$;

$1 \leq i \leq 12, +, \times_n\}$ be the SSS- pseudo linear algebra over the S-special interval pseudo ring R = [0, 14).

Study questions (i) to (vii) of problem 18 for this V.

24. Let $V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix} \right|$ $a_i \in [0, 39)$; $1 \leq i \leq 12\}$ be the

SSS- pseudo linear algebra over the S-special pseudo interval ring R = [0, 39).

(i) Define three distinct pseudo inner products on V.

(ii) Find $W^{\perp}$ given $W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ 0 \end{bmatrix} \right|$ $a_1, a_2 \in [0, 39) \} \subseteq V$

is a SSS- pseudo subspace of V.

25. Obtain some special and interesting features enjoyed by pseudo inner product on SSS- pseudo vector space over the S-special pseudo interval ring R = [0, n).

26. Let $M = \left\{ \left[\begin{array}{cc|cc|c} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & \dots & \dots & \dots & a_{10} \\ \hline a_{11} & \dots & \dots & \dots & a_{15} \\ a_{16} & \dots & \dots & \dots & a_{20} \\ \hline a_{21} & \dots & \dots & \dots & a_{25} \end{array} \right] \right|$ $a_i \in [0, 28);$

$1 \leq i \leq 25\}$ be a SSS-pseudo vector space over the S-special pseudo interval ring R = [0, 28).

(i) Define a pseudo inner product on V.

(ii) Define $f_{sss} : V \to [0, 28)$ and find V^*_{sss}.

(iii) Find a basis of V and V^*_{sss} (V^*_{sss} is the SSS-dual pseudo vector space of V the SSS-vector space).

27. Let $V = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ \hline a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ \hline a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ \hline a_{19} & a_{20} & a_{21} \\ a_{22} & a_{23} & a_{24} \\ a_{25} & a_{26} & a_{27} \\ a_{28} & a_{29} & a_{30} \end{array}\right] \middle| \; a_i \in [0, 17);\ 1 \leq i \leq 30 \right\}$

be the SSS- pseudo vector space over the S-special pseudo interval ring R = [0, 17).

Study questions (i) to (iii) of problem 26 for this V.

28. Let $V = \left\{ \left[\begin{array}{c|ccc|cc|cc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ a_9 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{16} \\ a_{17} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{24} \\ a_{25} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{32} \end{array}\right] \middle| \; a_i \in \right.$

$[0, 26); 1 \leq i \leq 32\}$ be the SSS- pseudo vector space over the S-special interval pseudo ring R = [0, 26).

Study questions (i) to (iii) of problem 26 for this V.

29. Let $V = \{(a_1, a_2, a_3, a_4 \mid a_5\ a_6\ a_7 \mid a_8\ a_9 \mid a_{10}) \mid a_i \in [0, 86); 1 \leq i \leq 10\}$ be the SSS- pseudo vector space over the S-special interval pseudo ring R = [0, 86).

Study questions (i) to (iii) of problem 26 for this V.

30. Let $V = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 23) \right\}$ be the SSS- pseudo vector space over the S- pseudo special interval ring R = [0, 23).

(i) Give some special properties enjoyed by this V.
(ii) Can V have finite dimensional SSS-pseudo vector subspaces?
(iii) Can V have infinite dimensional SSS-pseudo vector subspaces?
(iv) Give a basis of V.
(v) How many basis can V have?

31. Let $V = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 48) \right\}$ be a SSS- pseudo linear algebra over the S- pseudo special interval ring R = [0, 48).

Study questions (i) to (v) of problem 30 for this V.

32. Let $V = \left\{ \left[\begin{array}{c} a_1 \\ \hline a_2 \\ a_3 \\ a_4 \\ \hline a_5 \end{array} \right] \,\middle|\, a_i \in [0, 19); 1 \leq i \leq 5 \right\}$ be the SSS- pseudo vector space over the S- pseudo special interval ring R = [0, 19).

(i) Find V* of V.
(ii) What is the dimension of V^*?

(iii) If $V = \left\{ \left[\begin{array}{c} 1 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \end{array}\right], \left[\begin{array}{c} 0 \\ \hline 5 \\ 0 \\ 0 \\ \hline 0 \end{array}\right], \left[\begin{array}{c} 0 \\ \hline 0 \\ 3 \\ 0 \\ \hline 0 \end{array}\right], \left[\begin{array}{c} 0 \\ \hline 0 \\ 0 \\ 7 \\ \hline 0 \end{array}\right], \left[\begin{array}{c} 1 \\ \hline 0 \\ 0 \\ 0 \\ \hline 11 \end{array}\right] \right\}$ is a basis of V find the corresponding basis for V*.

(iv) Find a basis of Hom_R (V, V) over R.

(v) Define an inner product on V.

33. Let $V = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_9 \\ a_{10} & a_{11} & \ldots & a_{18} \\ a_{19} & a_{20} & \ldots & a_{27} \\ a_{28} & a_{29} & \ldots & a_{36} \\ a_{37} & a_{38} & \ldots & a_{45} \\ a_{46} & a_{47} & \ldots & a_{54} \end{bmatrix} \right| a_i \in [0, 43); 1 \leq i \leq 54\}$

be a SSS- pseudo vector space over the S-special pseudo interval ring R = [0, 43).

Study questions (i) to (v) of problem 32 for this V.

34. Let $V = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{12} \\ a_{13} & a_{14} & \ldots & a_{24} \\ a_{25} & a_{26} & \ldots & a_{36} \end{bmatrix} \right| a_i \in [0, 53); 1 \leq i \leq 36\}$

be a SSS- pseudo vector space over the S-special pseudo interval ring R = [0, 53).

Study questions (i) to (v) of problem 32 for this V.

35. Let $V = \left\{ \sum_{i=0}^{25} a_i x^i \,\middle|\, a_i \in [0, 41), 0 \leq i \leq 25 \right\}$ be a SSS-pseudo linear algebra over the S- pseudo special interval ring R = [0, 41).

    (i) Study questions (i) to (v) of problem 32 for this V.
    (ii) Show V is finite dimensional.
    (iii) Find a basis of V over R.

36. Let $V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_7 \end{bmatrix} \,\middle|\, a_i \in \left\{ \sum_{i=0}^{\infty} g_i x^i \,\middle|\, g_i \in [0, 46) \right\} \right\}$ be the SSS-pseudo vector space over the S- pseudo special interval ring R = [0, 46).

    Study questions (i) to (v) of problem 32 for this S.

37. Let $V = \left\{ (a_1, a_2, a_3, \ldots, a_9) \,\middle|\, a_i \in \left\{ \sum_{i=0}^{\infty} g_i x^i \,\middle|\, g_i \in [0, 29), 1 \leq i \leq 9 \right\} \right\}$ be the SSS- pseudo vector space over the S-pseudo special interval ring R = [0, 29).

    (i) Prove V is also a SSS- pseudo linear algebra.
    (ii) What is the dimension of V as a SSS- pseudo vector space over R?
    (iii) What is the dimension of V as a SSS- pseudo linear algebra over R?
    (iv) Find SSS- pseudo sublinear algebras.
    (v) Prove V has SSS-quasi vector spaces of finite dimension over R.

38. Let $V = \left\{ \left. \begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \\ a_{31} & a_{32} & \dots & a_{40} \end{bmatrix} \right| a_i \in \left\{ \left. \sum_{i=0}^{\infty} g_i x^i \right| g_i \in \right.\right.$ $[0, 93), 1 \leq i \leq 40\}$ be the SSS- pseudo linear algebra under natural product $\times_n$ over the S- pseudo special interval ring R = [0, 93).

Study questions (i) to (v) of problem 37 for this S.

39. Let $V = \left\{ \left. \begin{bmatrix} a_1 & a_2 & \dots & a_{16} \\ a_{17} & a_{18} & \dots & a_{32} \\ a_{33} & a_{34} & \dots & a_{48} \\ a_{49} & a_{50} & \dots & a_{64} \end{bmatrix} \right| a_i \in \left\{ \left. \sum_{i=0}^{\infty} g_i x^i \right| g_j \in \right.\right.$ $[0, 6), 1 \leq i \leq 64\}$ be the SSS- pseudo linear algebra under natural product $\times_n$ over R = [0, 6).

Study questions (i) to (v) of problem 37 for this M.

40. Let $V = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} \right| a_i \in \left\{ \left. \sum_{i=0}^{10} g_i x^i \right| g_i \in [0, 41), 1 \leq i \leq 9 \right\}\right.$ be the SSS-vector space over the S-pseudo special interval ring R = [0, 41).

(i) Prove V is not a SSS- pseudo linear algebra.
(ii) Is V finite dimensional?

(iii) Find a basis of V over R.
(iv) Find $\text{Hom}_R(V, V)$.
(v) Find V* of V.
(vi) Define a pseudo inner product on V.

(vii) Find for the set $A = \left\{ \begin{bmatrix} 9x+2 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\} \subseteq V$. $A^\perp$ is $A^\perp$ a SSS- pseudo subspace of V over R.

41. Let $M = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_9 \\ a_{10} & a_{11} & \ldots & a_{18} \\ a_{19} & a_{20} & \ldots & a_{27} \end{bmatrix} \middle| \; a_i \in \left\{ \sum_{i=0}^{12} g_i x^i \middle| \; g_j \in [0, 5), 0 \leq j \leq 12, 1 \leq i \leq 27 \right\} \right.$ be the SSS- pseudo vector space over the S- pseudo special interval ring R = [0, 5).

Study questions (i) to (vii) of problem 40 for this M.

42. Let $T = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \middle| \; a_i \in \left\{ \sum_{i=0}^{16} g_i x^i \middle| \; g_j \in [0, 15), 0 \leq j \leq 16, 1 \leq i \leq 30 \right\} \right.$ be the SSS- pseudo vector space over the S-special pseudo interval ring R = [0, 15).

Study questions (i) to (vii) of problem 40 for this T.

BeatriceGloria_personal library

43. Let $V = \{(a_1 \mid a_2\ a_3 \mid a_4\ a_5\ a_6 \mid a_7\ a_8\ a_9\ a_{10} \mid a_{11}\ a_{12}\ a_{13}\ a_{14}\ a_{15}$

$a_{16}) \mid a_i \in \left\{ \sum_{i=0}^{5} g_i x^i \,\middle|\, g_j \in [0, 7), 0 \leq j \leq 5, 1 \leq i \leq 16 \right\}$ be

the SSS-vector space over the S-special pseudo interval ring R = [0, 7).

Study questions (i) to (vii) of problem 40 for this V.

44. $$\text{Let } V = \left\{ \left[\begin{array}{c|cccc|cc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_8 & \cdots & \cdots & \cdots & \cdots & \cdots & a_{14} \\ a_{15} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{21} \\ \hline a_{22} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{28} \\ a_{29} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{35} \\ \hline a_{36} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{42} \\ a_{43} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{49} \\ \hline a_5 & \cdots & \cdots & \cdots & \cdots & \cdots & a_{56} \end{array} \right] \middle|\, a_i \in \right.$$

$\left\{ \sum_{i=0}^{3} g_i x^i \,\middle|\, g_j \in [0, 13), 0 \leq j \leq 3, 1 \leq i \leq 56 \right\}$ be the SSS-

pseudo vector space over the S-special interval pseudo ring R = [0, 13).

Study questions (i) to (vii) of problem 40 for this V.

45. Let $V = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ \hline a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ \hline a_{16} & a_{17} & a_{18} \\ \hline a_{19} & a_{20} & a_{21} \\ a_{22} & a_{23} & a_{24} \\ a_{25} & a_{26} & a_{27} \\ a_{28} & a_{29} & a_{30} \\ \hline a_{31} & a_{32} & a_{33} \\ a_{34} & a_{35} & a_{36} \end{array} \right] \middle| \; a_i \in \left\{ \sum_{i=0}^{2} g_i x^i \middle| \; g_j \in [0, 2), \right. \right.$

$0 \leq j \leq 7$, $1 \leq i \leq 36\}$ be the SSS-pseudo vector space over the S-special pseudo interval ring R = [0, 2).

(i) Study questions (i) to (vii) of problem 40 for this V.

(ii) If we put $x^8 = 1$ can V be made into a SSS- pseudo linear algebra under the natural product $\times_n$ of matrices.

(iii) Find dimension of V as a SSS-pseudo linear algebra over R.

FURTHER READING

1. Birkhoff. G, *Lattice theory*, 2nd Edition, Amer-Math Soc. Providence RI 1948.
2. Castillo J., *The Smarandache semigroup*, International Conference on Combinatorial Methods in Mathematics, II meeting of the project 'Algebra, Geometria e Combinatoria', Faculdade de Ciencias da Universidade do Porto, Portugal, 9-11 July 1998.
3. Padilla, Raul, *Smarandache algebraic structures*, Bull. Pure Appl. Sci., **17E**, 199-121 (1998).
4. Vasantha Kandasamy, W. B., *On zero divisors in reduced group rings over ordered rings*, Proc. Japan Acad., **60**, No.9, 333-334 (1984).
5. Vasantha Kandasamy, W. B., *A note on the modular group ring of a finite p-group*, Kyungpook Math. J., **26**, 163-166 (1986).
6. Vasantha Kandasamy, W.B., *Smarandache Rings*, American Research Press, Rehoboth, 2002.
7. Vasantha Kandasamy, W.B., *Smarandache Semigroups*, American Research Press, Rehoboth, 2002.
8. Vasantha Kandasamy, W.B., *Smarandache Semirings, Semifields and Semivector spaces*, American Research Press, Rehoboth, 2002.
9. Vasantha Kandasamy, W.B., *Linear Algebra and Smarandache Linear Algebra*, Bookman Publishing, US, 2003.

10. Vasantha Kandasamy, W.B. and Florentin Smarandache, *Set Linear Algebra and Set Fuzzy Linear Algebra*, InfoLearnQuest, Phoenix, 2008.
11. Vasantha Kandasamy, W.B. and Florentin Smarandache, *Algebraic Structures using [0, n)*, Educational Publisher Inc., Ohio, 2013.

INDEX

D

Direct sum of S-special linear subalgebras, 174-9

I

Infinite pseudo integral domain, 7-9

N

Non commutative special interval pseudo linear algebra, 70-6

P

Pseudo integral domain, 9-12

S

S- pseudo special interval ring, 9-13
Smarandache special interval vector space, 107-115
Special interval pseudo linear algebra, 44-8
Special interval pseudo polynomial vector space, 87-95
Special interval vector space, 9-13
Special pseudo interval integral domains, 84-8

Special vector spaces, 9-13
S-special idempotent linear operator, 207-9
S-special interval pseudo linear algebra, 107-112
S-special interval vector spaces, 7-9, 107-115
S-special linear operator, 142-9, 195-209
S-special linear projection, 201-8
S-special linear pseudo subalgebra, 110-9
S-special linear transformation, 142-9
S-special pseudo interval ring, 7-8, 222-215
S-special quasi vector subspace, 112-9
S-special strong pseudo interval vector space, 222-10
SSS- pseudo vector space, 7-8
SSS-characteristic value, 236-8
SSS-interval pseudo linear algebra, 220-9
SSS-interval pseudo linear subalgebra, 220-9
SSS-interval pseudo vector subspace, 220-9
SSS-linear functional, 239-242
SSS-special interval vector space, 222-10
Subsubdirect subspace sum, 30-4
Subvector quasi space, 44-50